Homework 7: Hausdorff spaces and Metrizability

"Modern mathematics rests on the substructure of mathematical logic and the theory of sets... Upon this base rise the two pillars that support the whole edifice: general algebra and general topology." - Lucienne Felix

1. Let X be compact and Hausdorff and let A be a closed subset of X. Define $\sim$ on X by $x \sim y$ iff $x = y$ or $x, y \in A$. Prove that $X/\sim$ is both compact and Hausdorff.

2. Let X be a Hausdorff space and let $f : X \to X$ be continuous. Define the fixed point set of f to be the set $F = \{x \in X | f(x) = x\}$. Prove that F is closed.

3. Let X be a set with topologies τ and ω. Suppose that (X, τ) is compact and (X, ω) is Hausdorff, and $\omega \subseteq \tau$. Prove that $\omega = \tau$.

4. Let X denote the product of uncountably many copies of $\mathbb{R}$. Prove that X is not metrizable.

5. We begin with some definitions. Let X be a topological space. A collection of subsets τ of X is said to be *locally finite* if for every $x \in X$ there is an open set U containing x such that U intersects only finitely many sets in τ. Let ω and τ both be collections of subsets of X, then τ is said to be a *refinement* of ω if for every $A \in \tau$ there is a $B \in \omega$ such that $A \subseteq B$. If all of the sets in τ are open then τ is said to be an *open refinement* of ω. Now X is said to be *paracompact* if it is Hausdorff and every open cover ω of X has a locally finite open refinement which also covers X.

Problem: Suppose that X is paracompact. Let $s \in X$ and let T be a closed set which does not contain s. Prove that there exist disjoint open sets U and W such that $s \in U$ and $T \subseteq W$.