Homework 8: Connectedness and Path Connectedness

"Unacountably, Nerzhin, the younger of the two men, the failure with no academic title, asked the questions, the creases around his mouth sharply drawn. And the older man answered as if he were ashamed of his unpretentious personal history as a scientist: evacuation in wartime, re-evacuation, three years of work with K--, a doctoral dissertation in mathematical topology. Nerzhin, who had become inattentive to the point of discourtesy, did not even ask Verenyov the subject of his dissertation in that dry science in which he himself had once taken a course. He was suddenly sorry for Verenyov. Quantities solved, quantities not solved, quantities unknowntopology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of some use to someone, but as for now ..." Aleksandr I. Solzhenitsyn

1. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

2. Let A be a connected subspace of a topological space X. Suppose that $A \subseteq Y \subseteq Cl(A)$. Prove that Y is connected.

3. Let $\sim$ be the equivalence relation on X defined by $x \sim y$ if and only if there is a path in X joining x and y. Determine whether $X/\sim$ being path connected implies that X is path connected.

4. Let X denote the square $I \times I$ with the dictionary topology. That is, open sets are unions of open intervals of the form $((a,b),(p,q))$ where $(a,b),(p,q) \in I \times I$ and $(a,b) < (p,q)$, together with intervals of the form $[(0,0),(p,q))$ and intervals of the form $((a,b),(1,1)]$. Prove that X is connected.

5. Prove that the space X in problem 4 is not path connected.