My teaching style:

- Proofs "In the round". Most students love it, but some find it too slow.
- Irrelevant questions. Most students love it, but some find it too irrelevant.
- I talk fast, but I'm happy to repeat.
- I expect a high level of rigor in proofs.
- If any of the above are really going to bother you, then you shouldn't take this class.

1. Basic Outline of Course

This course has three parts.

(1) Background material and examples of topological spaces.

(2) Using known spaces to construct new spaces.

(3) Using topological properties to distinguish spaces.

We begin with an introduction to Part (1).

2. Intuitive introduction to Topology

What is geometry? Geometry is the study of rigid shapes that can be distinguished with measurements (length, angle, area, . . .).

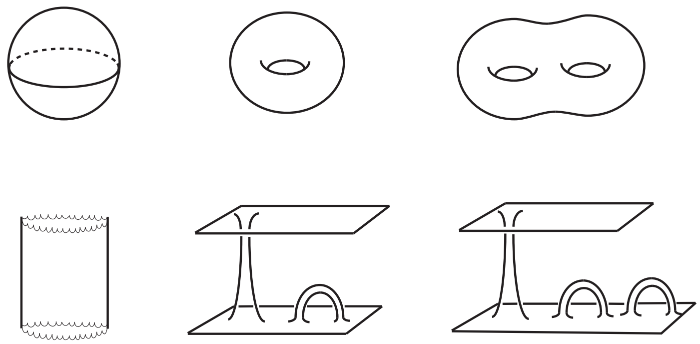
What is topology? Topology is the study of those characteristics of shapes and spaces which are preserved by deformations.

Topology versus Geometry: Objects that have the same topology do not necessarily have the same geometry. For instance, a square and a triangle have different geometries but the same topology. A line and a circle have different topologies, since one cannot be deformed to the other. The topology of a space tells us the "essential structure" of the space.

Motivation: We would like to know the topology of our universe and other possible universes. Locally, our universe looks like $\mathbb{R}^3$, but that doesn't mean that globally it's $\mathbb{R}^3$. Topologists would like to have a (infinite) list of all possible spaces that locally look like $\mathbb{R}^3$. But finding such a list is an open question.

Let's think about the analogous question in 2-D

What are some examples of two-dimensional universes? A plane, a sphere, a torus, and planes connected by one or more tubes.



These are topologically distinct universes. Adding a bump to one of these surfaces changes its geometry but not its topology. Intuitively, we can see that the number and type of "holes" is what distinguishes the topology of these surfaces. Hence we would like a mathematical way to describe holes. But this is not easy

3. Metric spaces

Recall the intuitive definition of continuity says that a function is continuous "if you can draw it without any gaps". This gives us the idea that the existence of holes has something to do with continuity. So we take the definition of continuity as the actual starting point for the course.

Question: Does anyone remember the definition of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$?

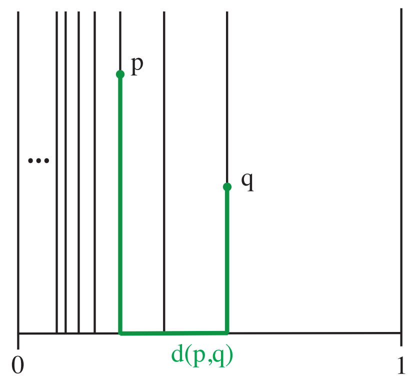
Since the definition of continuity makes sense for any space with a notion of distance, we might as well consider continuity of functions between metric spaces.

Question: Does anyone remember the definition of a metric space from Real Analysis?

In Analysis we saw examples of metric spaces. Euclidean space and the discrete metric are important examples that we will refer to. What's the discrete metric?

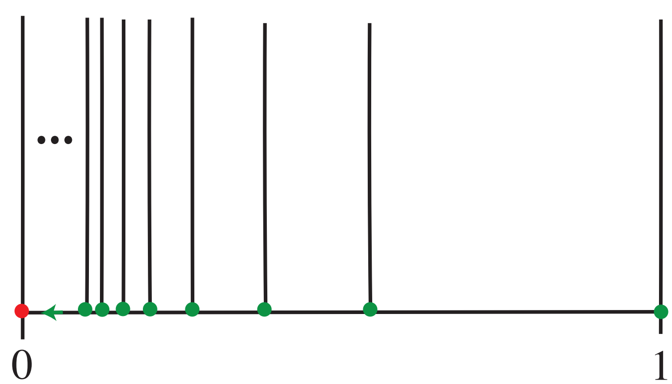
Now we consider a different example.

Example (The Comb Space.). Let $X_0 = \{0\} \times [0,1]$, $Y_0 = [0,1] \times \{0\}$; and $\forall n \in \mathbb{N}$, let $X_n = \{\frac{1}{n}\} \times [0,1]$. Let $M = (\bigcup_{n=0}^{\infty} X_n) \cup Y_0$ be "the comb". The metric we use is the distance measured along the comb in $\mathbb{R}^2$.



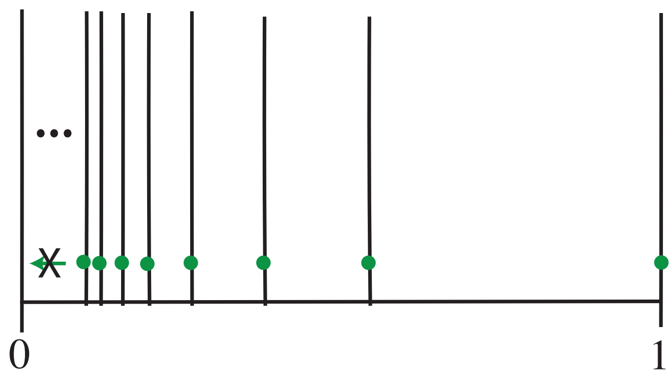
Using the comb metric,

- Does the sequence $\{(\frac{1}{n}, 0)\}$ converge in the comb metric space? Yes, to the origin.



- Does the sequence $\{(\frac{1}{n}, a)\}$ converge when $a \in (0,1]$?

  No. We can see that the sequence isn't Cauchy since for any $m, n \in \mathbb{N}$, we have $d((\frac{1}{n}, a), (\frac{1}{m}, a) > 2a$.

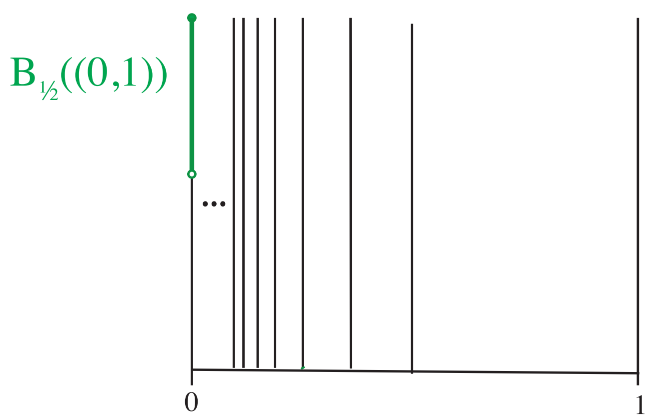


The fundamental tool that we use in studying metric spaces is the "open ball".

Question: Does anyone remember how we define an open ball in a metric space ?

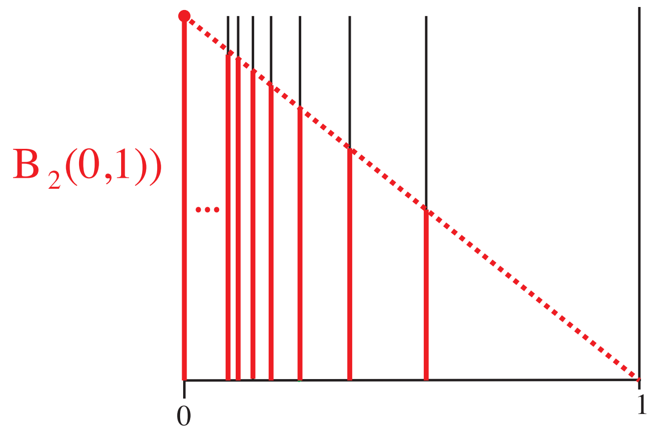
Question: In the comb space, what is $B_{\frac{1}{2}}((0,1))$?

$B_{\frac{1}{2}}((0,1))$ is just a vertical interval along the y-axis going down from $(0,1)$

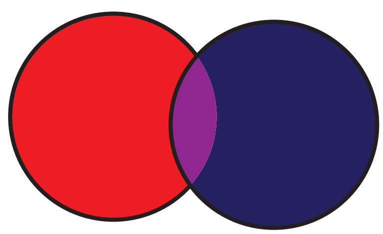


Question: In the comb space what is $B_2((0,1))$?

This is everything on the comb below the line of slope -1 that connects points $(0,1)$ and $(1,0)$. Since it's an open ball, the bounding line is not contained in the ball.



In Analysis we observed that balls aren't closed under $\bigcap$ and $\bigcup$, which makes them bad.



Hence we defined open sets.

Question: What's the definition of an open set in a metric space?

Recall the following Important theorems (Note the definition of "Important" in this class is that you will probably need this result on the homework):

THEOREM ("open sets behave well" theorem). *Let F be the family of open sets in a metric space (M, d). Then:*

(1) $M, \emptyset \in F$

(2) *If* $U, V \in F$ *then* $U \cap V \in F$

(3) *If* $\forall i \in I, U_i \in F$ *then* $\bigcup_{i\in I} U_i \in F$

THEOREM (Continuity in terms of open sets theorem). *Let M_1 and M_2 be metric spaces and $f : M_1 \to M_2$. Then f is continuous iff for every open set $U \subseteq M_2$, $f^{-1}(U)$ is open in M_1.*

From these two theorems we see that open sets are wonderful. In fact, if continuity is what we are after (to understand holes), we only need open sets, we don't need open balls or even a metric. So rather than defining open sets in terms of open balls, we just choose any collection of sets which is well behaved in terms of $\bigcup$ and $\bigcap$ and declare them to be our open sets.

4. Topological Spaces

Definition. Let X be a set and F be some collection of subsets of X such that

1) $X, \emptyset \in F$.

2) If $U, V \in F$ then $U \cap V \in F$.

3) If for all $i \in I$, $U_i \in F$, then $\bigcup U_i \in F$.

Then we say that (X, F) is a **topological space** whose **open sets** are the elements of F. We say F is the **topology** on X.

Note we are choosing the collection F of open sets for X. There is more than one choice of F for a given X. Just remember balls are not defined in an arbitrary topological space.

Example. Let (M, d) be any metric space and F be the set of open sets in M. Then (M, F) is a topological space.

Example. Let M be a set and d be the discrete metric, then we say M has the **discrete topology**. What sets are open in the discrete topology? We can also define the discrete topology without starting with a metric by saying every subset of M is open.

Example. Let X be a set with at least 2 points. Let $F = \{X, \emptyset\}$. Then we say (X, F) is the **indiscrete**, or **concrete** topology. Why do we require X to have at least 2 points?

Definition. If F_1 and F_2 are topologies on X and $F_1 \subseteq F_2$ then we say that F_1 is **weaker** than F_2, and F_2 is **stronger** than F_1.

This is hard to remember. So we have the following notes.

- weaker = smaller = coarser = fewer grains of sand (which are the open sets)
- stronger = bigger = finer = more grains of sand (which are the open sets)
- The discrete topology is the strongest topology on M. Why?
- The indiscrete topology is the weakest topology on X.

Example. Let $X = \mathbb{R}$ and $U \in F$ iff U is the union of sets of the form $[a, b)$ such that $a, b \in \mathbb{R}$ together with the empty set. (This is called the **half-open interval topology**).

Question: Is the half-open topology $(\mathbb{R}, F)$ weaker, stronger, or neither compared to the usual topology?

If every open interval (a, b) is in F, then F is stronger (i.e. bigger). Let $a, b \in \mathbb{R}$, and $a < b$. Then,

$$(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b),$$

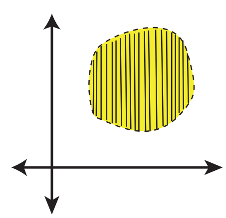
so F is stronger.

Example. We define the "dictionary order" on $X = \mathbb{R}^2$ by:

$(a, b) < (c, d)$ if either $a < c$ or $a = c$ and $b < d$.

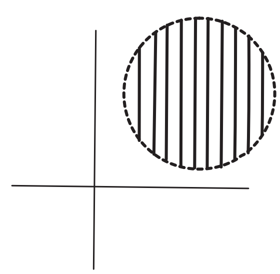
We define the dictionary topology F on $\mathbb{R}^2$ as $U \in F$ iff U is a union of "open intervals", of the form

$$U = \{(x, y) \mid (a, b) < (x, y) < (c, d)\}.$$



- Is a vertical line open? Yes.
- Is a horizontal line open? No.
- Is this topology finer or coarser or neither than the usual topology on $\mathbb{R}^2$?

  Any point in an open ball in the usual topology on $\mathbb{R}^2$ is contained in an open interval of the dictionary topology, which is contained in the ball. Thus open balls are open in the dictionary topology. Hence the dictionary topology is finer than the usual topology.



Question: Are the topologies on a set linearly ordered? No!

EXAMPLE. Consider $\mathbb{R}$ with the usual topology and $(\mathbb{R}, F)$ with $F = \{\mathbb{R}, \emptyset, \{47\}\}$. We say these two topologies are **incomparable**.

EXAMPLE. Consider $(\mathbb{R}, F)$ where $U \in F$ if and only if either $U = \emptyset$, $U = \mathbb{R}$, or $\mathbb{R} - U$ is finite. This topology is called the **finite complement topology on** $\mathbb{R}$. How does this topology compare with the usual topology?

We see that if U is open in $(\mathbb{R}, F)$, then U is open in the usual topology. Therefore the usual topology is finer than the finite complement topology.

DEFINITION. A set C in a topological space (X, F) is **closed** iff its complement $X - C$ is open.

Note that a set can be both open and closed (i.e., *clopen*) as well as neither open nor closed. (In particular, a set is not a door!)

Question: Is there a non-trivial clopen set in $\mathbb{R}$ with the half-open interval topology?

Yes, $[0, \infty) = \bigcup_{n \in \mathbb{N}} [0, n)$ is open. On the other hand, the complement of this set is $(-\infty, 0) = \bigcup_{n \in \mathbb{N}} [-n, 0)$, which is also open. Thus this is a **clopen** set.

Question: Is there a non-trivial clopen set in $\mathbb{R}^2$ with the dictionary order?

Yes, a vertical line

Question: Is there a non-trivial clopen set in $\mathbb{R}$ with the finite complement topology?

No, if U is clopen then so is $\mathbb{R} - U$. But if U is non-trivial then both U and $\mathbb{R} - U$ are finite.

LEMMA. *Let (X, F) be a topological space and A be the set of all closed sets in X. Then:*

(1) $X, \emptyset \in A$

(2) *If $C, D \in A$, then $C \cup D \in A$*

(3) *If $C_i \in A$ for every $i \in I$, then $\bigcap_{i \in I} C_i \in A$*

The proof follows from the definition of open sets together with the equations:

$$X - \bigcap_{i \in I} U_i = \bigcup_{i \in I} (X - U_i)$$

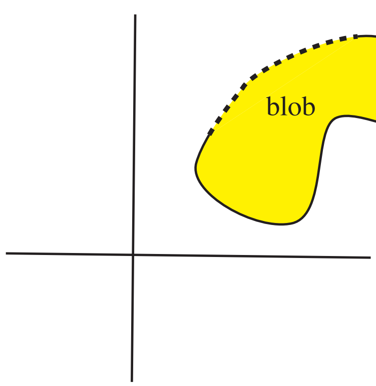
and

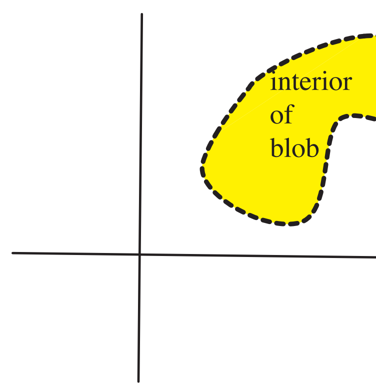
$$X - \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X - U_i)$$

We don't prove this because it's boring. Soon we will start doing proofs in the round.

Since open and closed sets behave well under unions and intersections, we would like to approximate arbitrary sets by open and closed sets. We will define interior and closure for this purpose.

DEFINITION. Let (X, F) be a topological space and $A \subseteq X$. Let $\{U_j \mid j \in J\}$ be the set of all open sets contained in A. Then we define $\mathring{A} = \operatorname{Int}(A) = \bigcup_{j \in J} U_j$, and we say $\mathring{A}$ is the **interior** of A.





Intuitively, $\mathring{A}$ is the "largest"open set contained in A. But what exactly do we mean by "largest"?

A does not contain a proper subset B which is open and contains $\mathring{A}$ as a proper subset.

SMALL FACT. Let (X, F) be a topological space and $A \subseteq X$. Then

(1) $\mathring{A} \subseteq A$

(2) $\mathring{A}$ is open

(3) If $U \subseteq A$ is open, then $U \subseteq \mathring{A}$

(4) A is open iff $A = \mathring{A}$.

PROOF. go around

(1) Since $\mathring{A} = \bigcup_{j \in J} U_j$ and $U_j \subseteq A$ for every $j \in J$, $\mathring{A} \subseteq A$.

(2) By definition, $\mathring{A}$ is a union of open sets, so $\mathring{A}$ is open.

(3) Since U is open in A, $U \in \{U_j \mid j \in J\}$. Therefore $U \subseteq \bigcup_{j \in J} U_j = \mathring{A}$.

Note that this means that $\mathring{A}$ is the "largest" open set in A.

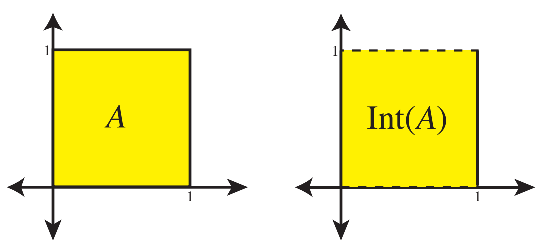
(4) Suppose that $A = \mathring{A}$. Then $\mathring{A}$ is open by part (2), hence A is open.

Conversely, suppose that A is open. Then $A \in \{U_j \mid j \in J\}$. Hence $A \subseteq \mathring{A}$. From (1), $A = \mathring{A}$. $\square$

Example. In the half-open interval topology on $\mathbb{R}$, $\mathrm{Int}((0,1]) = (0,1)$. To prove this, assume that $1 \in \mathrm{Int}((0,1])$ and show that it leads to a contradiction.

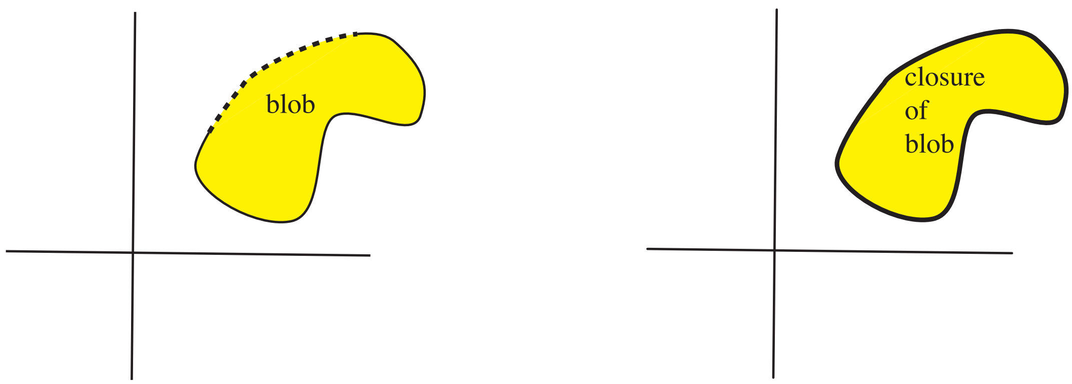
Example. In the finite-complement topology on $\mathbb{R}$, $\mathrm{Int}((0,1]) = \emptyset$. This follows from the fact that $\mathbb{R} - ((0,1])$ is infinite.

Example. In the dictionary-order topology on $\mathbb{R}^2$, $\mathrm{Int}([0,1] \times [0,1]) = [0,1] \times (0,1)$.



Next we want to approximate sets by closed sets.

Definition. Let A be a subset of a topological space (X, F), and let $\{F_j \mid j \in J\}$ be the set of all closed sets containing A. Then the **closure** of A is defined as $\overline{A} = cl(A) = \bigcap_{j \in J} F_j$.



Observe that $\overline{A}$ is the smallest closed set containing A.

Small Fact. Let (X, F) be a topological space and $A \subseteq X$. Then

(1) $A \subseteq \overline{A}$

(2) $\overline{A}$ is closed

(3) If $A \subseteq C$ and C is closed, then $\overline{A} \subseteq C$

(4) $\overline{A} = A$ if and only if A is closed.

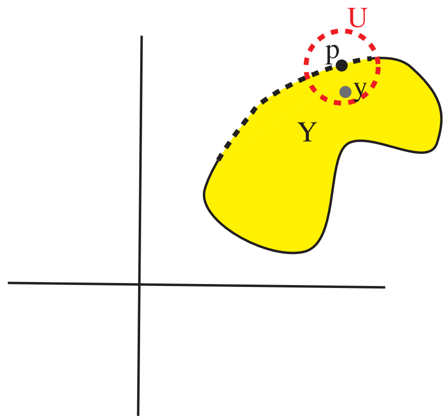
The proofs are left as exercises.

Example. In $\mathbb{R}$ with the usual topology, $\mathrm{cl}\left(\left\{\frac{1}{n} | n \in \mathbb{N}\right\}\right) = \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup \{0\}$

Example. In $\mathbb{R}$ with the half-open interval topology, $\mathrm{cl}((0,1]) = [0,1]$

Lemma (Important Lemma). *Let (X, F) be a topological space and $Y \subseteq X$. Then $p \in \overline{Y}$ if and only if for every open set $U \subseteq X$ containing p, $U \cap Y \neq \emptyset$.*

Note that Important Lemmas are results that should be used on the homework if you're stuck.



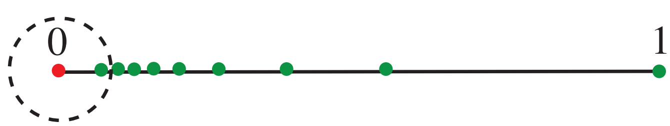
Proof. ($\Longrightarrow$) Let $p \in \overline{Y}$ and $U \subseteq X$ be open with $p \in U$. Suppose $U \cap Y = \emptyset$ and let $C = X - U$. Then $p \notin C$ because $p \in U$. Also, by the Small Facts $\overline{Y} \subseteq C$ because $Y \subseteq C$ and C is closed. Now since $p \in \overline{Y}$, we have $p \in C$ which is a contradiction.

($\Longleftarrow$) Suppose that for every open set $U \subseteq X$ such that $p \in U$, $U \cap Y \neq \emptyset$. In order to prove that $p \in \overline{Y}$, we need to show that p is in every closed set containing Y. So let $C \subseteq X$ be closed such that $Y \subseteq C$. Suppose $p \notin C$. Let $U = X - C$, which is open with $p \in U$. Now $U \cap Y \neq \emptyset$, so there exists some $x \in U \cap Y$. This means that $x \in U = X - C$, and hence $x \notin C$. But since $Y \subseteq C$, we have $x \notin Y$, giving us a contradiction. Therefore we conclude that $p \in C$ and hence $p \in \overline{Y}$. $\square$

Corollary. *Suppose that U is an open set in a topological space (X, F) and $Y \subseteq X$. If $U \bigcap \overline{Y} \neq \emptyset$, then $U \bigcap Y \neq \emptyset$.*

Proof. The proof is immediate from the Important Lemma.

Now we can use the Important Lemma to conclude that in $\mathbb{R}$ with the usual topology we have $\mathrm{cl}(\{1/n \mid n \in \mathbb{N}\}) = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Observe from the figure that for all open sets U containing 0, we have $U \bigcap \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \neq \emptyset$.



On the other hand this is not true for any point outside of $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$.

5. Continuity in Topological Spaces

Recall that we have the following theorem for metric spaces:

THEOREM. *Let* (M_1, d_1) *and* (M_2, d_2) *be metric spaces and* $f : M_1 \to M_2$. *Then* f *is continuous if and only if for every* $U \subseteq M_2$ *that is open in* (M_2, d_2), $f^{-1}(U)$ *is open in* (M_1, d_1).

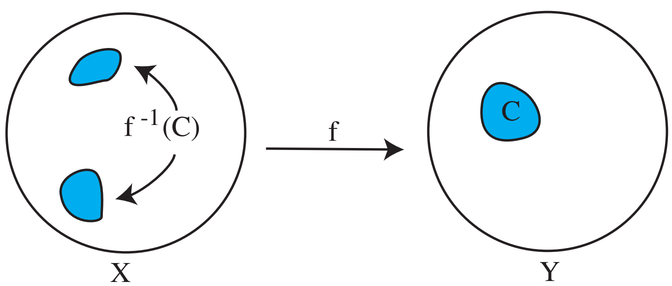
This motivates the following definition:

DEFINITION. Let (X_1, F_1) and (X_2, F_2) be topological spaces and $f : X_1 \to X_2$. We say f is **continuous** if and only if for every $U \in F_2$, $f^{-1}(U) \in F_1$.

SMALL FACT. Let (X_1, F_1) and (X_2, F_2) and (X_3, F_3) be topological spaces, $f : X_1 \to X_2$ and $f : X_2 \to X_3$ be continuous functions. Then $g \circ f : X_1 \to X_3$ is continuous.

PROOF. Let $U \in F_3$. Then $g^{-1}(U) \in F_2$ because g is continuous, and $f^{-1}(g^{-1}(U)) \in F_1$ because f is continuous. Therefore $(g \circ f)^{-1}(U) \in F_1$. Thus $g \circ f$ is continuous. □

THEOREM. *Let* X, Y *be topological spaces and* $f : X \to Y$. *Then* f *is continuous if and only if for every closed set* C *in* Y, $f^{-1}(C)$ *is closed in* X.



PROOF. Before we prove either direction, we prove the following set theoretic result.

Claim: For every set $A \subseteq Y$, we have $f^{-1}(Y - A) = X - f^{-1}(A)$.

Proof of Claim: $f^{-1}(Y - A) = \{x \in X \mid f(x) \in Y - A\} = \{x \in X \mid f(x) \notin A\} = X - \{x \in X \mid f(x) \in A\} = X - f^{-1}(A)$

($\Rightarrow$) Suppose f is continuous and C is closed in Y. Then $Y - C$ is open, implying that $f^{-1}(Y - C) = X - f^{-1}(C)$ is open in X. Hence $f^{-1}(C)$ is closed.

($\Leftarrow$) Suppose that for every closed set C in Y, $f^{-1}(C)$ is closed in X. Let U be open in Y. Now $f^{-1}(Y - U) = X - f^{-1}(U)$ is closed in X. Hence $f^{-1}(U)$ is open in X. □

This is nice, because sometimes it's easier to work with closed sets than with open sets.

Definition. Let X and Y be topological spaces, and let $f : X \to Y$.

(1) If for every open set $U \subseteq X$, $f(U)$ is open in Y, then we say f is **open**.

(2) If for every closed set $U \subseteq X$, $f(U)$ is closed in Y, then we say f is **closed**.

This is sort of like continuity, except that we care about the images of sets instead of their preimages.

Example. Let F be the half-open topology on $\mathbb{R}$, and define a function

$$f : (\mathbb{R}, F) \to (\mathbb{R}, \text{usual}) \qquad \text{by } f(x) = x$$

- Is f continuous? Yes! An open set in $(\mathbb{R}, \text{usual})$ is a union of intervals of the form (a, b). We know that $f^{-1}\big((a, b)\big) = (a, b)$, which is open in F.
- Is f open? No. Take any $U = [a, b) \in F$. Then $f(U) = [a, b)$, which is not open in $(\mathbb{R}, \text{usual})$.
- Is f closed? Also no, since $f\big([a, b)\big) = [a, b)$ is not closed in $(\mathbb{R}, \text{usual})$.

6. Homeomorphisms

Now we define what we mean by equivalence for topological spaces.

Definition. Let (X_1, F_1) and (X_2, F_2) be topological spaces, and let $f : X_1 \to X_2$ be continuous, bijective, and open. Then f is a **homeomorphism**, and X_1 and X_2 are **homeomorphic** (denoted by $X_1 \cong X_2$).

Example. Let $I = [0, 1]$ and $X = I \times I \subset \mathbb{R}^2$, under the usual metric for $\mathbb{R}^2$. Let $Y = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} = D^2$ be the unit disk in $\mathbb{R}^2$. Question: Are X and Y homeomorphic?

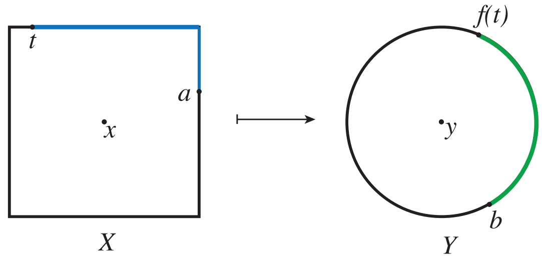
Answer: Yes! Let's see how.

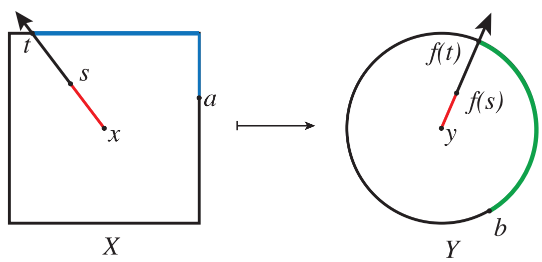
Define the centers of X and Y to be x and y, respectively. Fix some point a on the boundary of X, and some b on the boundary of Y (that is, $a \in \partial X$ and $b \in \partial Y$).

Define a function f as follows:

- $f(x) = y$
- $f(a) = b$
- For $t \in \partial X$, look at the distance along the boundary from a to t. Then $f(t)$ is a point proportionally far along the boundary of Y.
- For $s \in \text{int}(X)$, draw the ray connecting x and s. Let t be the point at which this ray intersects ∂X. Now, in Y, draw the ray connecting y and $f(s)$. Then s is mapped to a point on this ray that is proportionally far from y.

The last two, in pictures:





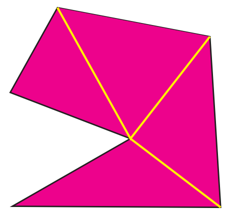
- Is this well-defined?
  - Yes! There is always exactly one point in Y that a point in X is mapped to[1].
- Is this a bijection?
  - Yes! The inverse is defined identically, so it would make sense for this to be a bijection. Also, consider the images of concentric squares centered on x under f: they are mapped to disjoint concentric circles centered on y.
- Is f continuous and open?
  - Yes! Intuitively, it's easy to see that an open set in X is mapped to an open set in Y, and that the preimage of an open set in Y is open.

This is good, since our intuition is that a square a circle should be the same topologically, since one can be deformed to the other.

Question: Can we extend this to (some) non-convex regions?

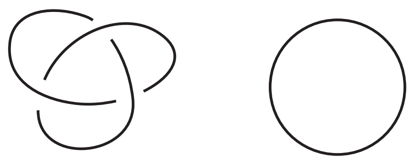
Answer: Sure. Just divide up the non-convex region into smaller regions. A subregion will work as long as there is some point in its interior such that any ray from that point intersects the subregion's boundary exactly once.

[1]This works because both a square and a circle are *convex* shapes - that is, for all points $p, q \in X$, the line $\overline{pq}$ that connects p and q lies entirely within X. This also implies that x and y didn't actually have to be the exact *centers* of X and Y respectively.



However, there are limits to this: for example, an annulus is *not* homeomorphic to a disk. This is hard to show; we'll see a proof later.

There are also some homeomorphisms that we might find unsettling. For example, a knot in $\mathbb{R}^3$ and the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ are homeomorphic; the argument works exactly like the one used for ∂X when showing that a square and circle are homeomorphic (above). It turns out that, while the knot and circle are homeomorphic, their complements are not, but the proof is beyond the scope of this class.

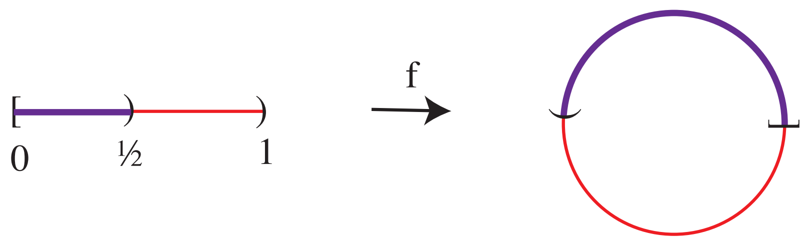


Example. Define a function $f : [0, 1) \to S^1$ by

$$f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$$

This takes the interval and wraps it counterclockwise around the circle.

- Is this a bijection?
  - Yes! The interval wraps once around the circle; one end is open, so there is no overlap.
- Is f continuous?
  - Yes! Recall that an open ball in S^1 is $B_\varepsilon(a) = \{x \in S^1 \mid d(x, a) < \varepsilon\}$. Then the preimage of every open ball in S^1 is an open interval, so f is continuous.
- Is f open?
  - No. Let $U = [0, 1/2)$, which is open in $[0, 1)$. Sadly, its image is not open in S^1.



So this f is not a homeomorphism.

We will show that $\mathbb{R}^2 \ncong \mathbb{R}^3$ eventually. This is difficult.

Example. Some non-examples of homeomorphisms:

- $\mathbb{Q}$ under the usual topology is not homeomorphic to $\mathbb{R}$ under the usual topology, since there is no bijection between $\mathbb{Q}$ and $\mathbb{R}$.
- $\mathbb{R}$ with the finite-complement topology (FCT) is not homeomorphic to $\mathbb{R}$ with the usual topology.

  Proof in the round:

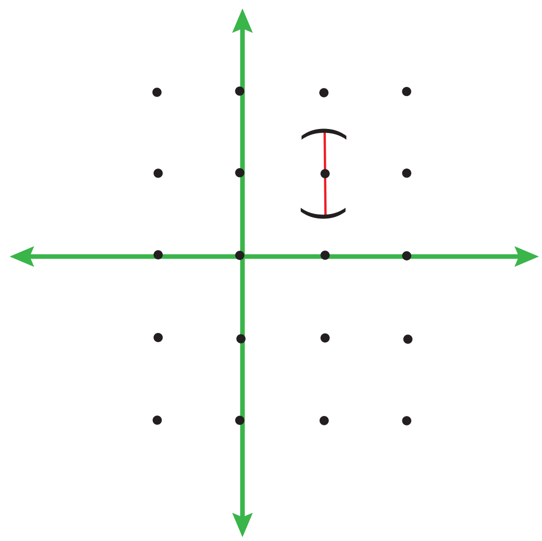
  - Suppose that there is some homeomorphism $f : (\mathbb{R}, FCT) \to (\mathbb{R}, \text{usual})$. Let $U = (0, 1)$.
  - Look at $f^{-1}(U)$ - it must be open, since f is a homeomorphism.
  - It is definitely not equal to $\emptyset$, since $U \neq \emptyset$.
  - It is also not $\mathbb{R}$, since f is a bijection.
  - So $\mathbb{R} - f^{-1}(U)$ must be finite. But $\mathbb{R} - U$ is not finite.
  - Because f is a bijection, this is a contradiction.
  - Therefore, $(\mathbb{R}, FCT) \ncong (\mathbb{R}, \text{usual})$.

7. Subspaces

Definition. Let (X, F_X) be a topological space, and let $Y \subseteq X$. Let $F_Y = \{U \cap Y | U \in F_X\}$. Then F_Y is the **subspace topology** or **induced topology** on Y.

Example. Consider the topological space $\mathbb{R}^2$ with the dictionary order.

Q: What are the open sets in the subspace $\mathbb{Z} \times \mathbb{Z}$?
A: All sets are open. We can draw a small open interval about any point, so the set of any one point is open, and any set is the union of such sets.



Because of all sets are open, this is the discrete topology on $\mathbb{Z} \times \mathbb{Z}$.

Example. Consider $\mathbb{Z} \times \mathbb{Z}$ as a subspace of $\mathbb{R}^2$, but this time with the usual topology. Again, all sets are open, as we can construct an open ball of radius $\frac{1}{2}$ about any point that doesn't intersect any others in the same way that we can construct an open interval. Thus again this is the discrete topology.

Small Fact. Let (X, F_X) and (Y, F_Y) be topological subspaces with (S, F_S) a subspace of X and (T, F_T) a subspace of Y. Then

(1) If $S \in F_X$, then $F_S \subseteq F_X$.

(2) A subset C is closed in S iff $\exists$ a closed set A in X such that $C = A \cap S$.

(3) Suppose $f : X \to Y$ is continuous. Then $f \mid S : S \to Y$ is continuous.

(4) Suppose $f : X \to Y$ is continuous and $f(X) \subseteq T$. Let $g : X \to T$ be defined by $g(x) = f(x)$ for every $x \in X$. Then g is continuous.

Proof. (1) If $S \in F_X$, then S is open in X. By definition $F_S = \{U \cap S : U \in F_X\} \subseteq F_X$

(2) ($\Rightarrow$) Let $C \subseteq S$ be closed. Because $S - C \in F_S$, there is some $U \in F_S$ such that $U \cap S = S - C$. Since U is open in X, we know that $X - U$ is closed in X.

We claim that the closed set we desire is $X - U$. Note that

$$(X - U) \cap S = (X \cap S) - (U \cap S) = S - (U \cap S) = S - (S - C) = C$$

and so we are done

($\Leftarrow$) Suppose that there is a closed set $A \subseteq X$ such that $C = A \cap S$. Then $X - A \in F_X$, so $(X - A) \cap S \in F_S$. But $(X - A) \cap S) = (X \cap S) - (A \cap S) = S - C$ is open in S, so C is closed in S.

(3) Let $U \in F_Y$. Since f is continuous, $f^{-1}(U) \in F_X$. Because $(f|S)^{-1}(U) = f^{-1}(U) \cap S$, $(f|S)^{-1}(U)$ is the intersection of an open set with S and so is open. Therefore, $f|S$ is continuous.

(4) Let $U \in F_T$. Then there is some $V \in F_Y$ such that $U = V \cap T$. Because $f : X \to Y$ is continuous, $f^{-1}(V) \in F_X$. Since $f(X) \subseteq T$,

$$f^{-1}(U) = f^{-1}(V \cap T) = f^{-1}(V)$$

Therefore, $f^{-1}(U)$ is open in X. But $f^{-1}(U) = g^{-1}(U)$, so $g^{-1}(U)$ is open and so g is continuous. $\square$

8. Bases

You may have noticed that, in metric spaces, the idea of open balls was quite useful. In particular, open balls have a standard form and any open set is just a union of open balls. We'd like to have a similar idea in topological spaces.

Definition. Let (X, F) be a topological space and $\beta \subseteq F$ such that for every $U \in F$, U is a union of elements in β. Then we say that β is a **basis** for F.

Note that a basis is not necessarily minimal (unlike a basis of a vector space), but it *is* more useful the more specific it is.

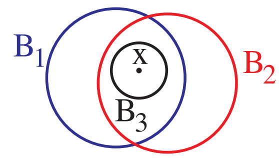
Example. $\mathbb{R}$ with the half-open interval topology was *defined* by the basis $\{[a, b) \mid a < b\}$.

Example. $\mathbb{R}^2$ with the dictionary topology was similarly defined by a basis of open intervals.

Theorem. *Let X be a set and β a collection of subsets of X such that*

(1) $X = \bigcup_{B \in \beta} B$

(2) *For all $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$, $\exists\ B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*



Let $F = \{$unions of elements of $\beta\} \cup \{\emptyset\}$. Then F is a topology for X with basis β.

Note, this theorem tells us that it is easy to construct a topology by defining a basis and just checking properties 1) and 2).

Proof. We prove that F is a topology for X by showing the three properties of a topological space as follows.

(1) Since $X = \bigcup_{B \in \beta} B$, by definition $X \in F$. We also have $\emptyset \in F$.

(2) Let $U, V \in F$. Hence we know that there exist index sets I and J such that $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} B_j$. Consider $U \cap V = (\bigcup_{i \in I} B_i) \cap (\bigcup_{j \in J} B_j)$. Let $x \in U \cap V$. Then there exists $i_x \in I$ and $j_x \in J$ such that $x \in B_{i_x} \cap B_{j_x}$. From our second assumption we know there exists a $B_x \in \beta$ such that $x \in B_x \subseteq B_{i_x} \cap B_{j_x}$. Let $W = \bigcup_{x \in U \cap V} B_x$. Since W is a union of elements of β it is clearly in F.

WTS: $W = U \cap V$

($\supseteq$) For all $x \in U \cap V$, we know that $x \in B_x \subseteq \bigcup_{x \in U \cap V} B_x = W$. Therefore $U \cap V \subseteq W$.

($\subseteq$) For all $y \in W$, $\exists x \in U \cap V$ such that $y \in B_x \subseteq B_{i_x} \cap B_{j_x} \subseteq U \cap V$. Therefore, $W \subseteq U \cap V$.

We have containment in both directions, so $W = U \cap V$.

(3) Suppose $\forall k \in K$, $U_k \in F$.

WTS: $\bigcup_{k \in K} U_k \in F$.

For all $k \in K$, $U_k = \bigcup_{i \in I_k} B_i$. Hence $\bigcup_{k \in K} U_k = \bigcup_{k \in K} (\bigcup_{i \in I_k} B_i) \in F$, since it is a union of elements of β.

Therefore F is a topology. By definition, β is also a basis of F. $\square$

SMALL FACT. Let (X, F_X) and (Y, F_Y) be topological spaces with bases of β_X and β_Y respectively, and $f : X \to Y$.

(1) f is continuous iff $\forall\ B \in \beta_Y,\ f^{-1}(B) \in F_X$.

(2) f is open iff $\forall\ B \in \beta_X,\ f(B) \in F_Y$.

PROOF. (of (1) only. (2) is virtually identical.)
($\Rightarrow$) Suppose f is continuous. Then $\forall\, U \in F_Y$, $f^{-1}(U) \in F_X$ by the definition of continuity. In particular, if $B \in \beta_y$, then $B \in F_Y$ and $f^{-1}(B) \in F_X$.

($\Leftarrow$) Suppose $B \in \beta_Y$ implies $f^{-1}(B) \in F_X$. Let $U \in F_Y$. Hence $U = \bigcup_{i\in I} B_i$ for some index set I. Therefore,

$$f^{-1}(U) = f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i) \in F_X,$$

since we know $f^{-1}(B_i) \in F_X$ for all i and unions of elements of F_X are in F_X. $\square$

TINY FACT (Important Tiny Fact about Bases). Let (X, F_X) be a topological space with basis β and let $W \subseteq X$. Then $W \in F_X$ iff $\forall p \in W$, $\exists B_p \in \beta$ such that $p \in B_p \subseteq W$.

This says that basis elements behave like balls.

PROOF. ($\Longrightarrow$) Let $W \in F_X$. Then $W = \bigcup_{i\in I} B_i$ with $B_i \in \beta$ for all $i \in I$. So $\forall p \in W$, $\exists B_p \in \beta$ such that $p \in B_p \subseteq W$. ✓

($\Longleftarrow$) Suppose $W \subseteq X$ such that $\forall p \in W$, $\exists B_p \in \beta$ such that $p \in B_p \subseteq W$. Then $W = \bigcup_{p\in W} B_p$. Hence $W \in F_X$. $\square$

Important Rule of Thumb. Using basis elements rather than arbitrary open sets often makes proofs easier.

Making new spaces from old

This is the beginning of Part 2 of the course.

9. Quotient Spaces

First we will consider quotients of sets.

DEFINITION. Let X be a set and $\sim$ a relation on X. We say $\sim$ is an **equivalence relation** if

(1) $\forall\ x \in X,\ x \sim x$ (**reflexivity**)

(2) $\forall\ x, y \in X$ if $x \sim y$ then $y \sim x$ (**symmetry**)

(3) If $x, y, z \in X$ and $x \sim y$ and $y \sim z$, then $x \sim z$ (**transitivity**)

DEFINITION. Let X be a set with an equivalence relation $\sim$. Then $\forall\ a \in X$ define the **equivalence class** of a as

$$[a] = \{x \in X : x \sim a\}.$$

DEFINITION. Let X be a set with an equivalence relation $\sim$. Then the **quotient** of X by $\sim$ is

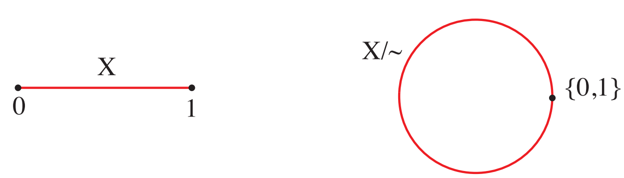
$$X/\sim = \{[p] : p \in X\}.$$

NOTE: The equivalence classes partition X. What does this mean? ($\forall\ x \in X$, $x \in$ exactly one equivalence class)

Question: give me an example of an equivalence relation other than $=$.

We want to think of starting with a set X, and then think of $X/\sim$ as what happens when equivalent points of X are glued together.

EXAMPLE. Let $X = [0, 1]$ and $x \sim y$ iff $x = y$ or $x, y \in \{0, 1\}$. This "glues" the interval $[0, 1]$ into a circle.



DEFINITION. Let X be a set and $\sim$ an equivalence relation. We define

$$\pi : X \to X/\sim$$

by $\pi(x) = [x]$. We say π is the **projection map**.

TINY FACT. Let X be a set with equivalence relation $\sim$. Then

(1) π is onto

(2) π is one-to-one iff "$\sim$" is "$=$".

PROOF. (1) Let $[x] \in X/\sim$. Then $\pi(x) = [x]$. Therefore π is onto.

(2) ($\Rightarrow$) Suppose π is one-to-one. Let $x, y \in X$ and $x \sim y$. Then $[x] = [y]$ and $\pi(x) = \pi(y)$, implying $x = y$ since π is one-to-one.
($\Leftarrow$) Suppose "$\sim$" is "$=$". Let $x, y \in X$ such that $\pi(x) = \pi(y)$. Then $\{x\} = [x] = [y] = \{y\}$, and $x = y$.

□

Now we would like to define a topology such that π is continuous.

Definition. Let (X, F_X) be a topological space and $\sim$ be an equivalence relation on X. We define

$$F_\sim = \{U \subseteq X/\sim : \pi^{-1}(U) \in F_X\}$$

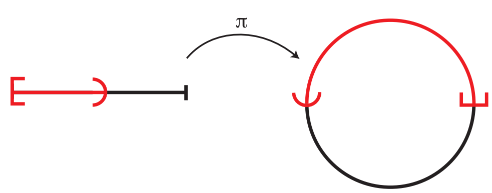
and call $(X/\sim, F_\sim)$ the **quotient space** of X with respect to $\sim$.

Example: Let $X = [0, 1]$ and $x \sim y$ iff $x = y$ or $x, y \in \{0, 1\}$. This "glues" the interval $[0, 1]$ into a circle.

Question: Is the quotient topology on the circle the same as the subspace topology induced by $\mathbb{R}^2$? Yes.

Question: Is π necessarily open? Answer: No

The interval [0,1/2), which is open in [0,1] is mapped to a half circle which is not open in the quotient space because it's inverse image contains the isolated point 1.



Small Fact. Let (X, F_X) be a topological space and $\sim$ be an equivalence relation on X. $(X/\sim, F_\sim)$ is a topological space and π is continuous.

Proof. First let us prove that $F_\sim$ is a topology on $X/\sim$.

(1) Since $\pi^{-1}(X/\sim) = X \in F_X$, we have $(X/\sim) \in F_\sim$. Since $\pi^{-1}(\emptyset) = \emptyset \in F_X$, we have $\emptyset \in F_\sim$.

(2) Let $U, V \in F_\sim$. Now

$$\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) \in F_X$$

since $\pi^{-1}(U) \in F_X$ and $\pi^{-1}(V) \in F_X$. Therefore $U \cap V \in F_\sim$.

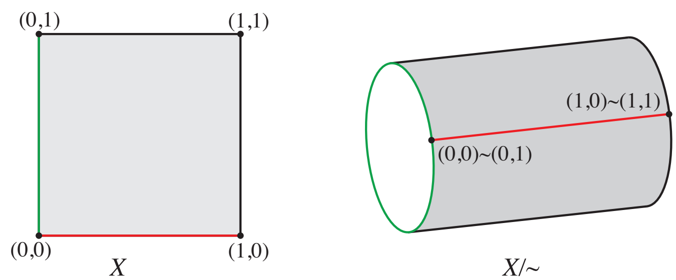
(3) Consider $\pi^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} \pi^{-1}(U_i)$.

Since for each i, $\pi^{-1}(U_i) \in F_\sim$, the arbitrary union of such sets must also be open. Thus by the above equality, $\pi^{-1}(\bigcup_{i\in I} U_i) \in F_\sim$, completing the proof.

Note that the continuity of π follows directly from the quotient topology.$\square$

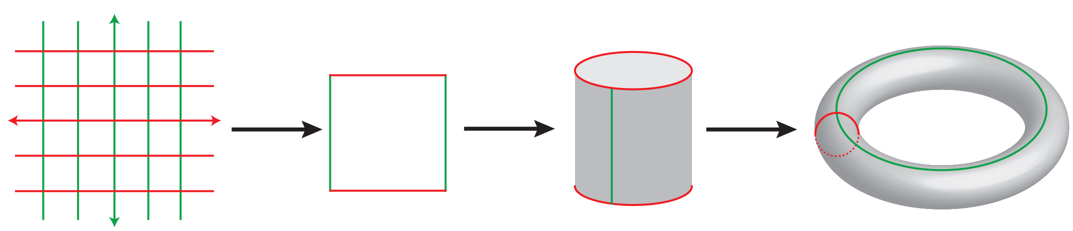
Example. Let $X = I \times I$, where I is the unit interval. Define an equivalence relation on X as follows: $(x, y) \sim (x', y')$ if and only if either $(x, y) = (x', y')$ or $x = x'$ and $y, y' \in \{0, 1\}$.

This equivalence "glues together" the top and bottom edges of the unit square. This basically rolls up the unit square, so topologically $X/\sim$ gives us a cylinder.



Example. Let $X = \mathbb{R}^2$ and suppose $(x, y) \sim (x', y')$ if and only if $\exists n, m \in \mathbb{Z}$ such that $x = x' + n, y = y' + m$.

Note that X may be divided into squares with integer sides such that under $\sim$ all of the given squares are equivalent. Thus we need only consider one such square, noting that the opposite sides are equivalent, yielding a torus. This is pretty much the same as the above example, except that we roll up the cylinder too. In the picture, we are identifying all lines of the same color.



We may also generalize this idea to higher dimensions, yielding the analogous torus for that dimension (i.e. $X = \mathbb{R}^3$ yields a 3-torus and so on).

In general, we say that a region is a *fundamental domain* if it is the smallest region such that gluing it up gives us the same quotient space as the quotient space obtained by gluing up the entire space. For example, the square in the above example. It's often useful to identify a fundamental domain in order to get an intuitive picture of what the quotient space looks like.

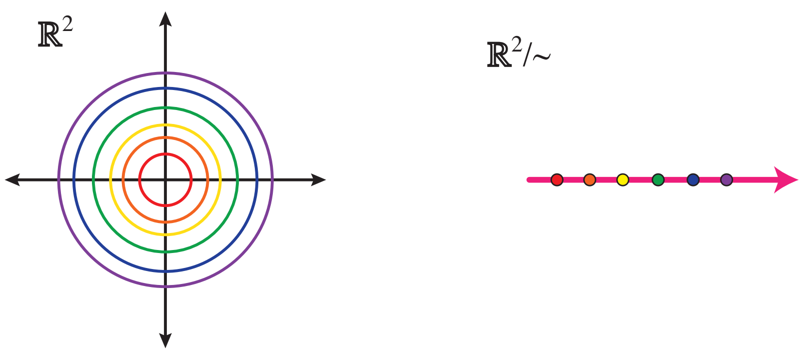
Example. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Define $x \sim y \Leftrightarrow x = \pm y$. We denote the quotient space $S^n/\sim$ by $\mathbb{RP}^n$. This space is called *real projective n-space.*

Note that for $n = 1$ we obtain the unit circle such that points connected by a diameter are equivalent. Thus any semi-circle forms a fundamental domain. Such a semicircle has it's endpoints as equivalent, and is thus topologically equivalent to the original circle. Thus $\mathbb{RP}^1 \cong S^1$.

From the above, $\mathbb{RP}^2 = S^2/\sim$. We simply note that the fundamental domain is a hemisphere whose boundary has opposite points glued.

Example. Let $X = \mathbb{R}^2$ and define $(x, y) \sim (x', y') \Leftrightarrow x^2 + y^2 = x'^2 + y'^2$.

Under $\sim$, any points on a circle centered at the origin are equivalent. Collapsing all such circles to points, we find that $X/\sim$ is a ray emanating from the origin (it's pink in the picture).



Example: Poincare Dodecahedral space could be our universe (bring a dodecahedron)

Now we consider a different approach to quotient spaces which **starts with a function rather than an equivalence relation**.

Definition. Let X, Y be sets and $f : X \to Y$ be onto. We define the relation $\sim$ **induced by** f as follows:

$$\forall p, q \in X, p \sim q \Leftrightarrow f(p) = f(q)$$

It is clear that $\sim$ is an equivalence relation on X because $=$ is an equivalence relation on Y.

Since f induces $\sim$, We could say that f induces $X/\sim$ But we would like to go directly from a function to the quotient space, without having to mention the equivalence relation. To do this we let f play the role of π. Then define a topology on Y to make f continuous.

Definition. Suppose (X, F_X) is a topological space and Y is a set. Let $f : X \to Y$ be onto. Then the **quotient topology** on Y with respect to f is given by:

$$F_f = \{U \subseteq Y | f^{-1}(U) \in F_X\}$$

We say that f is a **quotient map** from (X, F_X) to (Y, F_f).

Tiny Fact. (1) (Y, F_f) is a topological space.

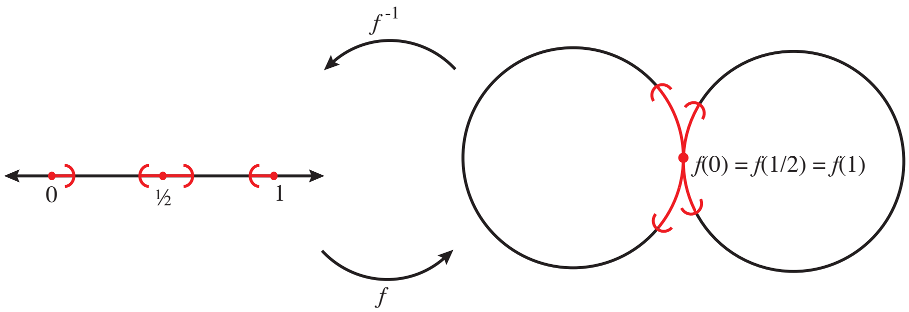
(2) $f : (X, F_X) \to (Y, F_f)$ is continuous.

(3) The quotient topology is the strongest topology on Y such that f is continuous.

Proof is left as an exercise.

Example. Let f map the closed segment $[0, 1]$ to a figure-eight, where $f(0) = f(1) = f(\frac{1}{2})$ at the intersection of the two sides of the figure-eight. What is open in the quotient topology (Y, F_f)?

- Any open-looking interval on the eight that does not contain the intersection point is open since its preimage is clearly open on the segment; for the same reason, any appropriate union or intersection of these open intervals is also open.
- Furthermore, any open set in the figure-eight that includes the point at the cross must also contain an open interval of nonzero length extending along each of the four legs of the figure-eight. This is necessary because the definition of F_f requires that the preimage of anything open must be open itself; hence the only way for the preimage of a set containing the intersection point to be open is if the preimage is an open set in $[0, 1]$ containing the three preimages of the intersection point (0, $\frac{1}{2}$, and 1).



The following theorem tells us that our two approaches to quotient spaces are equivalent.

Theorem. *Let* (X, F_X) *be a topological space and let* Y *be a set,* $f : X \twoheadrightarrow Y$ *onto, and* $\sim$ *induced by* f*. Then* $(Y, F_f) \cong (X/\sim, F_\sim)$*.*

We say a *diagram commutes* if the path taken does not affect the result. Note that the arrow representing g is dashed because we have not yet defined g. In the proof, we will define g so that the diagram does commute, and then prove that g is a homeomorphism.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & X/\sim \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Proof. Define $g\colon X/\sim \to Y$ by $g([x]) = f(x)$ where x is a representative of the equivalence class $[x]$. We need to show that f is well-defined, one-to-one, onto, continuous, and open.

- Well-defined: WTS if $[x] = [y]$, then $g([x]) = g([y])$. That is, any representative of a given equivalence class maps to the same value. Suppose $[x] = [y]$. Then $x \sim y$. Now since $\sim$ is the relation induced by f, we have $f(x) = f(y)$. Now by definition of g we have $g([x]) = g([y])$.
- One-to-one: Suppose $g([x]) = g([y])$. Then $f(x) = f(y) \Rightarrow x \sim y \Rightarrow [x] = [y]$.
- Onto: Suppose $y \in Y$. Since f is onto, we know $\exists x \in X$ such that $f(x) = y$. So $g([x]) = f(x) = y$.
- Continuous: WTS $U \in F_f \Rightarrow g^{-1}(U) \in F_\sim$. Suppose $U \in F_f$. Recalling that $F_\sim = \{O \subseteq X/\sim \,|\pi^{-1}(O) \in F_X\}$, to show that $g^{-1}(U) \in F_\sim$, we need to show that $\pi^{-1}(g^{-1}(U)) \in F_X$. Recall that by definition of F_f we have $U \in F_f$ $\Leftrightarrow f^{-1}(U) \in F_X$. Thus we know that $f^{-1}(U) \in F_X$. If we could show that

$f^{-1}(U) = \pi^{-1}(g^{-1}(U))$, then we would know that $\pi^{-1}(g^{-1}(U)) \in F_X$. Note since since f and π are not bijections, this is not obvious. We show

$$f^{-1}(U) = \pi^{-1}(g^{-1}(U))$$

by showing containment in both directions.

($\subseteq$) Let $x \in f^{-1}(U)$. Hence $f(x) \in U$. WTS that $x \in \pi^{-1}(g^{-1}(U)) = \{p \in X | g \circ \pi(p) \in U\}$. But $g \circ \pi(x) = g([x]) = f(x) \in U$, and hence $x \in \pi^{-1}(g^{-1}(U))$.

($\supseteq$) Let $x \in \pi^{-1}(g^{-1}(U))$. Hence $g(\pi(x)) \in U \Rightarrow g([x]) \in U \Rightarrow f(x) \in U$. So $x \in f^{-1}(U)$.

$\therefore$ $f^{-1}(U) = \pi^{-1}(g^{-1}(U))$. Since $f^{-1}(U) \in F_X$, $\pi^{-1}(g^{-1}(U)) \in F_X$. Thus $g^{-1}(U) \in F_\sim$.

- Open: WTS $U \in F_\sim \Rightarrow g(U) \in F_f$. Suppose $U \in F_\sim$. Recalling that $F_f = \{O \subseteq Y | f^{-1}(O) \in F_X\}$, to show $g(U) \in F_f$, we need to show that $f^{-1}(g(U)) \in F_X$. Since $U \in F_\sim \Leftrightarrow \pi^{-1}(U) \in F_X$, we would be done if we could show

$$f^{-1}(g(U)) = \pi^{-1}(U)$$

We again show containment in both directions as follows.

($\subseteq$) Let $x \in f^{-1}(g(U))$. So $g([x]) = f(x) \in g(U)$. Since g is bijective (from the earlier parts of this proof), it follows that $[x] \in U$. Now we have $\pi(x) = [x] \in U$, and hence $x \in \pi^{-1}(U)$.

($\supseteq$) Let $x \in \pi^{-1}(U)$. So $\pi(x) \in U$, implying that $g(\pi(x)) \in g(U)$. Now $g(\pi(x)) = g([x]) = f(x)$ is in $g(U)$. Thus it follows that $x \in f^{-1}(g(U))$.

Therefore g is a homeomorphism, and $(Y, F_f) \cong (X/\sim, F_\sim)$. $\square$

This theorem means that starting with f and using f to define $\sim$, we get the same quotient space up to homeomorphism as we would if we had just used f to define the quotient space directly. Now we want to go the other way. Suppose we start with $\sim$ and use $\sim$ to define f, will we again get the same quotient space with respect to f and $\sim$? But this is easy to prove.

Tiny Fact. Let (X, F_X) be a topological space with an equivalence relation $\sim$. Let $Y = X/\sim$ and let $f : X \to Y$ be the projection map π. Then $F_\sim = F_f$ and f is a quotient map.

Proof. Recall that:

$$F_\sim = \{U \subseteq X/\sim \text{ st. } \pi^{-1}(U) \in F_X\}$$
$$F_f = \{U \subseteq Y \text{ st. } f^{-1}(U) \in F_X\}.$$

We know that $X/\sim = Y$ and $f = \pi$. $\therefore$ $F_\sim = F_f$. Also, $f = \pi$ is onto since it is a projection map, and by definition $F_\sim = F_f$ is a quotient topology. $\therefore$ f is a quotient map. $\square$

From now on when we talk about quotient spaces we use equivalence relations or quotient maps interchangeably depending on what's most convenient.

Lemma. *(Important Lemma about Quotients)*
Let (X, F_X), (Z, F_Z) be topological spaces and let Y be a set. Suppose $f : X \twoheadrightarrow Y$ is onto and let $g : (Y, F_f) \to (Z, F_Z)$. Then g is continuous if and only if $g \circ f$ is continuous.

Note that for any equivalence relation $\sim$ on X we could replace f by $\pi : X \to X/\sim$ to conclude that $g : (X/\sim, F_\sim) \to (Z, F_Z)$ is continuous if and only if $g \circ \pi$ is continuous.

The following summarizes the relationship between the different maps in the lemma. Note we don't have any dotted lines and by definition the diagram commutes.

$$\begin{tikzcd}
(X, F_X) \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & (Y, F_f) \arrow[dl, "g"] \\
& (Z, F_Z) &
\end{tikzcd}$$

Proof. $(\Rightarrow)$: Suppose that g is continuous. Since f is a quotient map, f is continuous. Therefore, $g \circ f$ is a composition of continuous functions, and so $g \circ f$ is continuous.

$(\Leftarrow)$: Suppose, that $g \circ f$ is continuous. Let $U \in F_Z$. Then $(g \circ f)^{-1}(U) \in F_X$. Therefore, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in F_X$.

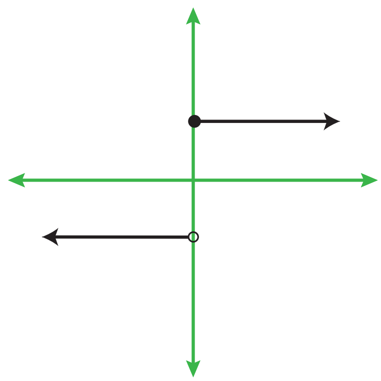
Recall that $F_f = \{V \subseteq Y : f^{-1}(V) \in F_X\}$. So, $V = g^{-1}(U) \in F_f$, and hence g is continuous. $\square$

Example (A Non-Example). Let $f : \mathbb{R} - \{1\} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 & x < 1 \\ \frac{-1}{x-1} & x > 1 \end{cases}$$

f is continuous and onto (why?). Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$



Then g is not continuous (why not?). But $g \circ f : \mathbb{R} - \{1\} \to \mathbb{R}$ is defined by

$$(g \circ f)(x) = \begin{cases} -1 & x > 1 \\ 1 & x < 1 \end{cases}$$

and $g \circ f$ is continuous.

Question: Does this contradict the Important Lemma?

Answer: No! The topology on the domain of g is not the quotient topology with respect to f, since $[0, 1)$ is not open in $\mathbb{R}$ which is the domain of g. But $f^{-1}([0, 1)) = (-1, 1)$ is open in $\mathbb{R}$. Thus we can't apply the Important Lemma.

For our next theorem, we would like to say that if we put anologous equivalence relations on homeomorphic spaces then the quotient spaces will be homeomorphic. For example, we obtain a torus by gluing up opposite sides of a square, but we should also get a torus if we glue up opposite sides of a trapezoid in an analogous way. The following theorem tells us that we do.

Theorem. *Let (A, F_A) and (B, F_B) be topological spaces and $f : A \to B$ be a homeomorphism. Let $\sim_A$ and $\sim_B$ be equivalence relations on A and B, respectively, such that $x \sim_A x'$ if and only if $f(x) \sim_B f(x')$.*

Then $A/\sim_A \cong B/\sim_B$.

Proof. We want to show that there is a function g which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi_A"'] & B \arrow[d, "\pi_B"] \\ A/\sim_A \arrow[r, dashed, "g"'] & B/\sim_B \end{tikzcd}$$

Define $g : A/\sim_A \to B/\sim_B$ by $g([x]_A) = [f(x)]_B$. To show it's a homeomorphism:

- **Well-defined:** Suppose $[y]_A = [z]_A$. Then
$$y \sim_A z \quad \Rightarrow \quad f(y) \sim_B f(z) \quad \Rightarrow \quad [f(y)]_B = [f(z)]_B$$
and so g is well-defined.

- **1-to-1:** Suppose that $[x]_A$ and $[y]_A$ are such that $g([x]_A) = g([y]_A)$. Then
$$[f(x)]_B = [f(y)]_B \quad \Rightarrow \quad f(x) \sim_B f(y) \quad \Rightarrow \quad x \sim_A y \quad \Rightarrow \quad [x]_A = [y]_A$$
and so g is 1-to-1.

- **Onto:** Let $[y]_B \in B/\sim_B$. Since f is onto, there is some $x \in A$ such that $f(x) = y$. Then
$$g([x]_A) = [f(x)]_B = [y]_B$$
Hence g is onto.

- **Continuous:** By the Important Lemma, g is continuous iff $g \circ \pi_A$ is continuous, since π_A is a quotient map. So we just need to show that $g \circ \pi_A$ is continuous. By the definition of g, $g \circ \pi_A = \pi_B \circ f$. Since f is a homeomorphism, it is continuous; and since π_B is a quotient map, f is continuous. So $\pi_B \circ f$ is continuous. Since $g \circ \pi_A = \pi_B \circ f$, we know that $g \circ \pi_A$ is continuous. Thus g is continuous.

- **Open:** Since f and g are both bijections, f^{-1} and g^{-1} are both functions. Instead of showing that g is open we will show that g^{-1} is continuous. We again want to use the Important Lemma to do this. First observe that
$$\begin{aligned} g \circ \pi_A &= \pi_B \circ f \\ g^{-1} \circ (g \circ \pi_A) \circ f^{-1} &= g^{-1} \circ (\pi_B \circ f) \circ f^{-1} \\ \pi_A \circ f^{-1} &= g^{-1} \circ \pi_B \end{aligned}$$
From here, the argument is eerily similar to the argument for continuity above.

  Note that π_A is a quotient map and f^{-1} is a homeomorphism, so both are continuous. As such, $\pi_A \circ f^{-1}$ is continuous, and therefore so is $g^{-1} \circ \pi_B$. Since π_B is a quotient map, by the Important Lemma, g^{-1} is continuous. Therefore, g is open.

Therefore, g is a homeomorphism. Hurrah! $\square$

10. The Product Topology

For the past several lectures, we've been building new topological spaces from old ones by using equivalence relations to form quotient spaces. Here, we will change directions and build new topological spaces from old ones by taking products of spaces. To that end, we wish to define the product of two sets:

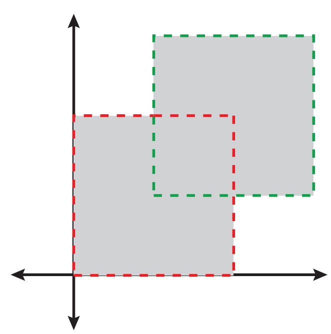
Definition. Let X, Y be sets. The product of X and Y is given by:

$$X \times Y \equiv \{(x, y) : \, x \in X, \, y \in Y\}$$

Note that a given point in a product $X \times Y$ cannot have two different representations as (x, y). Why not?

This is a topology class, so our intrinsic urge is to find a natural topology for the product of two topological spaces. While our first instinct might be to take products of open sets in our original spaces, this approach will give unsatisfactory results:

Example. Consider the sets $A = (0, 1) \times (0, 1)$ and $B = (1/2, 3/2) \times (1/2, 3/2)$ as subsets of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with the usual topology. Then $A \cup B$ in $\mathbb{R}^2$ is NOT a product of an open set in $\mathbb{R}$ with an open set in $\mathbb{R}$ as we would like. To intuitively see that this is not the case, see the picture! On the other hand, $A \cup B$ is open in the usual topology on $\mathbb{R}^2$.



Although products of open sets will not work because they are not closed under unions, we can use products of open sets to define a basis.

Definition. Let (X, F_x) and (Y, F_y) be topological spaces. Then the set $\beta_{X \times Y}$ is defined by:

$$\beta_{X \times Y} = \{A \times B : \, A \in F_x, \, B \in F_y\}$$

and define:

$$F_{X \times Y} = \left\{ \bigcup_{i \in I} U_i | \, I \text{ is some index set, and } U_i \in \beta_{X \times Y} \right\}$$

In other words, $\beta_{X\times Y}$ is the set of products of an open set in X and an open set in Y, and $F_{X\times Y}$ is the set of unions of elements in $\beta_{X\times Y}$. The motivation for defining our sets this way is that we want $F_{X\times Y}$ to be a topology on $X \times Y$ and for β to be its basis. We will now verify this claim with the following *small fact*:

Small Fact. If (X, F_x) and (Y, F_y) are topological spaces, the space $(X \times Y, F_{X\times Y})$ is a topological space with basis $\beta_{X\times Y}$.

Proof. We will apply our basis theorem; i.e. we want to show that

(1) $\displaystyle\bigcup_{U\in\beta_{X\times Y}} U = X \times Y$

(2) Given $B_1, B_2 \in \beta_{X\times Y}$, for each $x \in B_1 \cap B_2$, there exists $B_3 \in B_{X\times Y}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For the first statement, we know that $X \in F_x$ and $Y \in F_y$ by definition of a topology so that $X \times Y \in \beta_{X\times Y}$. It follows then that because each $U \in \beta_{X\times Y}$ is subset of $X \times Y$:

$$X \times Y \subseteq \bigcup_{U\in\beta_{X\times Y}} U \subseteq X \times Y$$

so that $X \times Y = \bigcup_{U\in\beta_{X\times Y}} U$, as desired.

For the second statement, let $B_1, B_2 \in \beta_{X\times Y}$ and let $(x, y) \in B_1 \cap B_2$. Then by definition of $\beta_{X\times Y}$, there exist $U_1, U_2 \in F_x$ and $V_1, V_2 \in F_y$ such that $B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2)$.

Our aim is to find $B_3 \in \beta_{X\times Y}$ such that B_3 contains (x, y) and $B_3 \subseteq B_1 \cap B_2$, so define:

$$B_3 = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Thus $U_1, U_2 \in F_x \Rightarrow U_1 \cap U_2 \in F_x$ by the closure of topologies under finite intersections and similarly, $V_1 \cap V_2 \in F_y$, so $B_3 \in \beta_{X\times Y}$. Since $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$, then $(x, y) \in (U_1 \times V_1)$ and $(x, y) \in (U_2 \times V_2)$. Therefore, $x \in U_1, U_2$ and $y \in V_1, V_2$, so $x \in U_1 \cap U_2$, and $y \in V_1 \cap V_2$. It immediately follows by the definition of the intersection of sets that:

$$(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2) = B_3$$

The only thing left to show is that $B_3 \subseteq B_1 \cap B_2$. To that end, let $(a, b) \in B_3$. Then $a \in U_1 \cap U_2$ and $b \in V_1 \cap V_2$ by definition of B_3. It follows that:

$a \in U_1, U_2$ and $b \in V_1, V_2 \Rightarrow (a, b) \in U_1 \times V_1$ and $(a, b) \in U_2 \times V_2 \Rightarrow a \in (U_1 \times V_1) \cap (U_2 \times V_2)$.

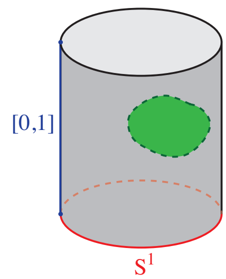
Therefore, $(a, b) \in B_1 \cap B_2$ so that $B_3 \subseteq B_1 \cap B_2$ because (a, b) is an arbitrary element of B_3.

Therefore, $\beta_{X\times Y}$ satisfies the hypotheses of our basis theorem, so the set of unions of elements of $\beta_{X\times Y}$, $F_{X\times Y}$, is a topology for $X \times Y$ and $\beta_{X\times Y}$ is a basis for the topology $F_{X\times Y}$. $\square$

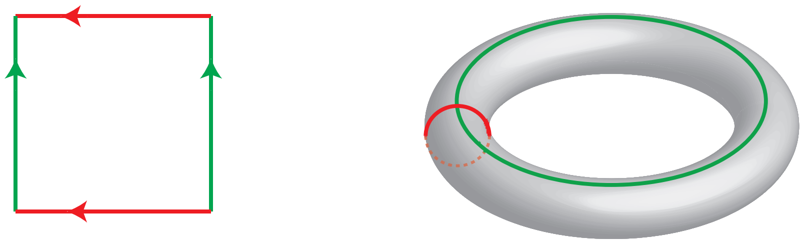
We have successfully devised a topology for product spaces as unions of products of open sets.

10.1. Examples.

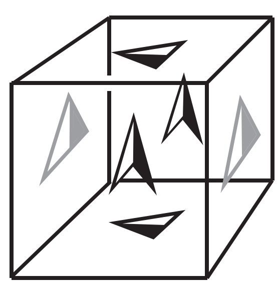
Example. The set $S^1 \times [0,1]$ looks like a cylinder! What kinds of sets are open in the cylinder? For example why is an open disc projected on the face of the cylinder open?



Example. The set $S^1 \times S^1$ looks like a torus! What kinds of sets are open in the torus? Similar to the previous example, open discs projected onto the torus surface are examples of open sets in $S^1 \times S^1$. In the picture, one copy of S^1 is red and one is green; they determine a torus.



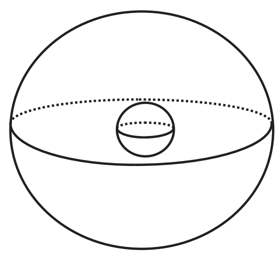
Example. The set $S^1 \times S^1 \times S^1$ is a 3-torus.



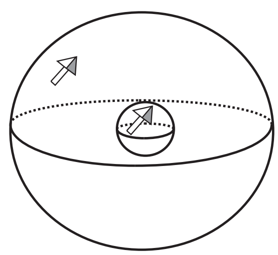
A *trivial product* is a product of a space X with a single point p. The reason we say that $X \times \{p\}$ is trivial is because it's homeomorphic to X. So taking the product has not created anything new.

Not every space is a non-trivial product. For example the sphere S^2 is not a non-trivial product. An intuitive (non-rigorous) justification is that the natural axes of a sphere are the great circles; however, every pair of distinct great circles intersect twice which makes it impossible to define a coordinate system on the sphere. Remember points in products have unique representations as a coordinate pair.

Example: $S^2 \times I$ looks like a chocolate Easter egg. That is, it's a thickened sphere.



Example: To get $S^2 \times S^1$ we start with $S^2 \times I$ and we glue the inside sphere to the outside sphere.



This could be the shape of our universe. How is it different from a 3-dimensional torus?

10.2. The Product Projection Map. Now that we have a product topology, we want a way to relate the product topology to the topology of the factors. In order to do so, we will define the projection maps as follows.

Definition. Let (X, F_x) and (Y, F_y) be topological spaces and create $X \times Y$ endowed with the product topology $F_{X \times Y}$. Define $\pi_X : (X \times Y, F_{X\times Y}) \to (X, F_x)$ and $\pi_Y : (X \times Y, F_{X\times Y}) \to (Y, F_y)$ by:

$$\pi_X((x,y)) = x \qquad \pi_Y((x,y)) = y.$$

The map π_X is the projection onto X and π_Y is the projection onto Y.

A small fact which we will derive now is that the product projection maps are continuous:

Small Fact. Let (X, F_X) and (Y, F_Y) be topological spaces and $(X \times Y, F_{X\times Y})$ be their product with the induced product topology. Then the projection maps π_X and π_Y onto X and Y are continuous.

Proof. Suppose $O \subseteq X$ and $O \in F_x$. We see that $\pi_X^{-1}(O) = O \times Y \in F_{X\times Y}$. Therefore, the pre-image of any open set in X under π_X is open in $X \times Y$ with the product topology. A similar argument shows that π_Y is continuous. $\square$

Recall that the quotient projection map is *not necessarily* an open map. It turns out that the product projection map *is* an open map. Accidentally assuming that the quotient map is open is a very common mistake that one should be aware of! We will now prove that the product projection map is open:

Tiny Fact. Let (X, F_X) and (Y, F_Y) be topological spaces and $(X \times Y, F_{X\times Y})$ be their product with the induced product topology. Then the projection maps π_X and π_Y onto X and Y are open.

Proof. Let $O = U \times V$ such that $U \in F_x$ and $V \in F_y$ and $O \in \beta_{X\times Y}$. Therefore:

$$\pi_X(O) = U \in F_x$$

Once we know that π_X takes basic open sets to open sets, it follows that π_X takes any open set to an open set. $\square$

Now that we have product spaces and have addressed their basic topological properties, we would like a way to easily find and construct continuous maps to the product space. To that end we introduce the following:

Lemma (Important Lemma about Products). *Let $(X, F_X), (Y, F_Y)$ and (A, F_A) be topological spaces and let $(X \times Y, F_{X\times Y})$ be the product space with the induced product topology. Suppose $f : A \to X$ and $g : A \to Y$ and define*

$$h : A \to (X \times Y) \qquad \textit{by} \qquad h(a) = (f(a), g(a)),$$

then h is continuous if and only if f, g are continuous.

Proof. ($\Rightarrow$) Suppose h is continuous: then we see that:

$$f = \pi_X \circ h \qquad g = \pi_Y \circ h$$

so f, g are continuous because they are compositions of continuous functions. ✓

($\Leftarrow$) Suppose that f and g are continuous. We prove that the preimage of every basis element under h is open. This will show that h is continuous.

Let $U \times V \in \beta_{X\times Y}$. Then $U \in F_X$ and $V \in F_Y$. Since f and g are continuous, $f^{-1}(U) \in F_A$ and $g^{-1}(V) \in F_A$. Then

$$\begin{aligned} h^{-1}(U \times V) &= \{a \in A \mid h(a) \in U \times V\} \\ &= \{a \in A \mid (f(a), g(a)) \in U \times V\} \\ &= \{a \in A \mid f(a) \in U, g(a) \in V\} \\ &= f^{-1}(U) \cap g^{-1}(V) \end{aligned}$$

Then $h^{-1}(U \times V)$ is the intersection of two open sets, hence it is open. Therefore, h is continuous. □

The following example illustrates how this lemma makes it very easy to define continuous functions to the product space:

EXAMPLE. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 + 3x$ and $g(x) = \sin(x)$. Then the map $h : \mathbb{R} \to \mathbb{R}^2$ defined by $h(x) = (x^2 + 3x, \sin(x))$ is continuous because f, g are.

10.3. Infinite Products. We would of course like to be able to generalize products to infinite products. Initially, we may be inclined to define such products as: $X_1 \times X_2 \times \cdots = \{(x_1, x_2, ...)|x_i \in X_i \forall i \in \mathbb{N}\}$. However, such a definition limits us to countable products, so we make the following definition.

DEFINITION. $\forall j \in J$, let X_j be a set. Define the product $\prod_{j\in J} X_j = \{f : J \to \bigcup_{j\in J} X_j | f(j) \in X_j\}$. We refer to $f(j)$ as the j^{th} coordinate of the point f.

In case you didn't notice, this definition is confusing.

EXAMPLE. Suppose $J = \{1, 2\}$. Then $\prod_{j\in\{1,2\}} X_j = \{f : \{1, 2\} \to X_1 \cup X_2 | f(j) \in X_j\} = \{(f(1), f(2))|f(j) \in X_j\} = \{(x_1, x_2)|x_j \in X_j\} = X_1 \text{ x } X_2$. Thus we see that this definition agrees with our previous definition for finite products.

EXAMPLE. Consider $\prod_{j\in\mathbb{R}}\{1, 2\}$. By definition this is $\{f : \mathbb{R} \to \{1, 2\}|f(j) \in \{1, 2\}, j \in \mathbb{R}\}$. This corresponds to subsets of $\mathbb{R}$, if we think of the pre-image of 1 under f as the elements in the set and the preimage of 2 under f as the elements not in the set. This product is denoted by $\{1, 2\}^{\mathbb{R}}$.

Now we wish to define a topology on products which agrees with our prior definition for a product of two sets if the indexing set $J = \{1, 2\}$. Perhaps the most natural way of doing this is the following.

DEFINITION. For each $j \in J$, let X_j have topology F_j. The *box topology* on $\prod_{j\in J} X_j$ is given by the basis

$$\beta_\Box = \{\prod_{j\in J} U_j | U_j \in F_j\}.$$
$$F_\Box = \{\text{unions of elements of } \beta_\Box\}.$$

While this topology makes sense for finite index sets, it does not work so well in general (as we will see in the proof of the Important Lemma below). But first we define another topology on infinite products.

Definition. For each $j \in J$, let X_j have topology F_j. The *product topology* on $\prod_{j\in J} X_j$ is given by the basis $\beta_\Pi = \{\prod_{j\in J} U_j | U_j = X_j$ for all but finitely many j, $U_j \in F_j\}$.

Remarks:

(1) β_Π and $\beta_\Box$ both agree with the product topology for $X_1 \times X_2$.

(2) $\beta_\Pi \subseteq \beta_\Box$

(3) Both are bases for topologies on the product.

(4) Unless explicitly stated otherwise, all products will be assumed to have the product topology.

Definition. Define the projection map $\pi_j : \prod_{j\in J} X_j \to X_j$ by $\pi_j(f) = f(j)$. So the j^{th} projection map takes a point to its j^{th} coordinate.

Lemma. *For all $j \in J$, let (X_j, F_j) be a topological space. Then*

$$\pi_{j_0} : \prod_{j\in J} X_j \to X_j \to X_{j_0}$$

is continuous for all $j \in J$.

Proof. Let $j_0 \in J$ and consider $U \in F_{j_0}$. We wish to show that $\pi_{j_0}^{-1}(U) \in F_\pi$. Note that $\pi_{j_0}^{-1}(U) = \{f \in \prod_{j\in J} X_j | \pi_{j_0}(f) \in U\} = \{f \in \prod_{j\in J} X_j | f(j_0) \in U\} = \{f \in \prod_{j\in J} X_j | f(j_0) \in U,\ \forall j \neq j_0,\ f(j) \in X_j\} = \prod_{j\in J} U_j$ such that $U_{j_0} = U$ and $\forall j \neq j_0, U_j = X_j$.

It then follows from the definition of the product topology that $\pi_{j_0}^{-1}(U) \in \beta_\pi \subseteq F_\pi$. So the projection map is continuous. $\Box$

The following Lemma is a generalization of the Important Lemma we had for continuous functions into products of two spaces. Does anyone remember what that lemma said?

Lemma (Important Lemma for Infinite Products). *Let (Y, F_Y) be a topological space, and for all $j \in J$ let (X_j, F_j) be a topological space and $g_j : Y \to X_j$. Define $h : Y \to \prod_{j\in J} X_j$ to be $h(y) = f \in \prod_{j\in J} X_j$ where f is defined by $\forall j \in J, f(j) = g_j(y)$. (i.e., for each j, the j^{th} coordinate of $h(y)$ is $g_j(y)$.) Then h is continuous if and only if every g_j is continuous.*

PROOF. ($\Rightarrow$) Suppose h is continuous and let $j \in J$. Then $\pi_j \circ h(y) = g_j(y)$. Thus g_j is the composition of continuous functions and must itself be continuous.

($\Leftarrow$) Suppose that g_j is continuous for all $j \in J$. Let $U \in \beta_\pi$. We wish to show that $h^{-1}(U) \in F_Y$.

Note that $h^{-1}(U) = \{y \in Y | h(x) \in U\}$. Since U is a basis element of the product topology, $U = \prod_{j \in J} U_j$ where $U_j \in F_j$ and $U_j = X_j$ for all but at most finitely many j.

$$h^{-1}(U) = \{y \in Y | h(y) \in \prod_{j \in J} U_j\} = \{y \in Y | g_j(y) \in U_j,\ \forall j \in J\}$$

$$= \{y \in Y | y \in g_j^{-1}(U_j),\ \forall j \in J\} = \bigcap_{j \in J} g^{-1}(U_j)$$

Since g_j is continuous for all j, $g_j^{-1}(U_j)$ is open in Y for all j. Since $U_j = X_j$ for all but at most finitely many j, $g_j^{-1}(U_j) = Y$ for all but at most finitely many j. It follows that the intersection $\bigcap_{j \in J} g^{-1}(U_j)$ is a finite intersection of open sets. Thus, $h^{-1}(U) \in F_Y$ so h is continuous, completing the proof. $\square$

Remark: Observe that this proof would not have worked if $\prod_{j \in J} X_j$ had the box topology rather than the product topology. This illustrates why the product topology is better than the box topology.

Distinguishing Spaces

DEFINITION. A **topological property** (or 'top. prop.') is a property of a topological space that is preserved by homeomorphisms.

The purpose of a topological property is to prove that two spaces are different. In particular, if one space has the property and the other one doesn't, then the spaces cannot be homeomorphic.

List of Topological Properties thus Far

(1) Cardinality of X

(2) Cardinality of F_x (Why?)

(3) Metrizability (we proved this in the homework)

(4) Discreteness (we'll prove this below)

(5) Indiscreteness (we'll prove this below)

LEMMA. *Let (X, F_X) and (Y, F_Y) be topological spaces and $f : X \to Y$ be an open bijection. If F_X is the discrete topology, then F_Y is the discrete topology.*

Note that we don't even need f to be continuous to reach this conclusion.

PROOF. Let $y \in Y$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Since $\{x\} \in F_X$ and f open, $f(\{x\}) \in F_Y$. Thus all singletons in Y are elements of F_Y and all sets in Y are unions of singletons and hence elements of F_Y, so every set is an open set and F_Y is the discrete topology. $\square$

LEMMA. *Let (X, F_X) and (Y, F_Y) be topological spaces and $f : X \to Y$ be a continuous bijection. If F_X is the indiscrete topology, then F_Y is the indiscrete topology.*

PROOF. Let $U \in F_Y$. WTS $U = Y$ or $U = \emptyset$. Since f is continuous, $f^{-1}(U) \in F_X$. Therefore $f^{-1}(U) = X$ or $\emptyset$. Suppose $f^{-1}(U) = X$. Then $U = f(f^{-1}(U)) = f(X) = Y$ since f surjective. Suppose $f^{-1}(U) = \emptyset$. Then $U = \emptyset$, again because f is onto. Therefore F_Y is the indiscrete topology. $\square$

Unfortunately, even with these five lovely Topological Properties, we can't yet distinguish a circle from a line. Clearly there is more work to do.

EXAMPLE (a non-example). Distance is not a topological property. A big circle *is* homeomorphic to a little circle.

What is the definition of **compact** for metric spaces? Recall, that the continuous image of a compact set is compact. The definition and proof of this result are the same for topological spaces. So compactness is a topological property. We won't go over compactness in topological spaces, because it is quite similar to compactness in metric spaces. Feel free to use results in the book about compact spaces. In particular, you may want to use the following three results:

THEOREM. *A closed subset of a compact space is compact.*

THEOREM (Bolzano-Weierstrass). *Let (X, F_X) be compact, and S be an infinite subset of X. Then X has a point p, such that every open set containing p contains infinitely many points of S.*

THEOREM (Tychonoff Theorem). *The product of finitely many compact spaces is compact.*

Note this is actually true for any finite or infinite collection of compact spaces.

11. Hausdorffness

DEFINITION. Let (X, F_X) be a topological space. We say that X is **Hausdorff** if $\forall\, p, q \in X$ such that $p \neq q$, $\exists$ disjoint sets $U, V \in F_X$ such that $p \in U$, $q \in V$.

EXAMPLE. Metric spaces **are** Hausdorff. Why?

Example (a non-example). Any space containing at least two points with the indiscrete topology is **not** Hausdorff. Why not?

Hausdorff is important because any space you would ever want to live in or work in is Hausdorff.

Note: on HW 4 you proved that the continuous image of a Hausdorff space need not be Hausdorff. Here is another Example.

Example. Define $f\colon (\mathbb{R},\ \text{discrete}) \to (\mathbb{R},\ \text{indiscrete})$ by the identity map.

This is continuous because the domain has the discrete topology, but the domain is Hausdorff while the range is not.

Small Fact. Suppose $f\colon (X, F_X) \to (Y, F_Y)$ is a homeomorphism and (X, F_X) is Hausdorff. Then (Y, F_Y) is also Hausdorff.

Proof. Let $p \neq q \in Y$. Because f is a bijection, $f^{-1}(p)$ and $f^{-1}(q)$ are defined and are distinct points in X. Since X is Hausdorff, $\exists\ U, V \in F_X$ such that $U \cap V = \emptyset$, $f^{-1}(p) \in U$, $f^{-1}(q) \in V$.

Consider $f(U)$ and $f(V)$. Since f is a homeomorphism and, thus, open, $f(U)$ and $f(V)$ are open. Then, because f is a bijection, we have $p \in f(U)$, $q \in f(V)$, and $f(U) \cap f(V) = \emptyset$. Hence indeed Y is Hausdorff. □

The relationship between Compact and Hausdorff. ♡

Hausdorffness and compactness are like two people in love. When two people love each other, they can do much more together than either one can alone.

Theorem. *Let (X, F_X) be Hausdorff. Let $A \subseteq X$ be compact. Then A is closed.*

Proof. (Compare the following proof to the proof from 131 that compact subsets of metric spaces are closed. That is, we are going to show that we don't need a metric, just Hausdorffness, for compact subsets to be closed.)

Rather than show A is closed, we will show $X - A$ is open.

Let $p \in X - A$. We want to show $\exists\ U \in F_X$ such that $p \in U \subseteq X - A$. Let $a \in A$. Because X is Hausdorff, $\exists\ U_a, V_a \in F_X$ such that $p \in U_a$, $a \in V_a$, $U_a \cap V_a = \emptyset$.

Consider $\{V_a | a \in A\}$. This is an open cover of A because each V_a is open and $\forall a \in A$, $a \in V_a$. So, because A is compact, $\{V_a | a \in A\}$ has a finite subcover $\{V_{a_1}, V_{a_2}, ..., V_{a_n}\}$.

Let $U = \bigcap_{i=1}^{n} U_{a_i}$. Then $U \in F_X$ since it is a finite intersection of elements of F_X and $p \in U$ since $p \in U_{a_i}\ \forall\ i = 1, 2, ..., n$.

Claim: $U \subseteq X - A$
Proof. $\forall\ i = 1, 2, ...\ n$, we have $U_{a_i} \cap V_{a_i} = \emptyset$. Also, $A \subseteq \bigcup_{i=1}^{n} V_{a_i}$ and $U \cap A \subseteq U \cap \bigcup_{i=1}^{n} V_{a_i}$.

However, $U \cap \bigcup_{i=1}^n V_{a_i} = \emptyset$ because if $\exists\, x \in U \cap \bigcup_{i=1}^n V_{a_i}$, then $\exists\, i_o$ such that $x \in V_{a_{i_o}} \cap U_{a_{i_o}}$. But, by definition, $V_{a_{i_o}} \cap U_{a_{i_o}} = \emptyset$. So no such x exists. Thus,

$$U \cap A \subseteq U \cap \bigcup_{i=1}^{n} V_{a_i} = \emptyset,$$

implying that $U \cap A = \emptyset$. So, because $U \subseteq X$ but $U \cap A = \emptyset$, $U \subseteq X - A$

Thus, we have found $U \in F_X$ such that $p \in U$, $U \subseteq X - A$. Thus, $X - A$ is open. Thus, **A is closed**. $\square$

Corollary. *In any Hausdorff space, (finite sets of) points are closed sets.*

Proof. Let $p \in (X, F_X)$ and let (X, F_X) be Hausdorff. Let $\{U_i | i \in I\}$ be an open cover of $\{p\}$. Then $\exists\, i_o \in I$ such that $p \in U_{i_o}$. Thus $\{U_{i_o}\}$ is a finite subcover. Hence, $\{p\}$ is compact. Thus, by the Theorem above, $\{p\}$ is closed. $\square$

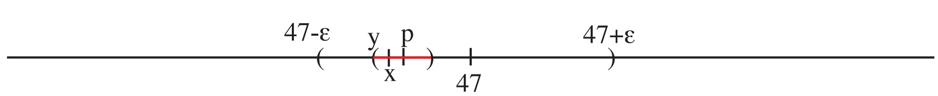
The following is a top prop like Hausdorff but stronger.

Definition. Let (X, F_x) be a topological space. We say X is **normal** if for every pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

On Homework 2, you proved that metric spaces are normal.

Example: A space that is Hausdorff but not normal. Consider $\mathbb{R}$ with a topology F defined by a basis of all sets of the form (a, b) plus all sets of the form $(a, b) \cap \mathbb{Q}$. This is known as the **rational topology**, and it is finer than the usual topology. As an exercise, you can prove that that this collection of sets really form the basis of a topology. (Use the Basis Lemma.)

$(\mathbb{R}, F)$ is Hausdorff because F contains the usual topology on $\mathbb{R}$, which is Hausdorff. Next, we will show that $(\mathbb{R}, F)$ is not normal. To see why, let $A = \mathbb{R} - \mathbb{Q}$. Then A is closed, because $\mathbb{Q} \in F$. Let $B = \{47\}$. Then B is closed because its complement is open. Suppose there exist $U, V \in F$ such that $A \subseteq U$, and $B \subseteq V$. Then there exists $\varepsilon > 0$ such that $(47 - \varepsilon, 47 + \varepsilon) \cap \mathbb{Q} \subseteq V$. Let $p \in (47 - \varepsilon, 47 + \varepsilon)$ such that $p \notin \mathbb{Q}$. Then $p \in U$ because $p \notin \mathbb{Q}$. So, there exists $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq U$.



Now, let $y = \max\{p - \delta, 47 - \epsilon\}$. Then there is a rational x such that $y < x < p$. Now we have

$$p - \delta \leq y < x < p < p + \delta$$

So $x \in (p - \delta, p + \delta) \subseteq U$. Also we have

$$47 - \epsilon \leq y < x < p < 47 + \epsilon$$

and $x \in \mathbb{Q}$. Thus $x \in (47 - \epsilon, 47 + \epsilon) \cap \mathbb{Q} \subseteq V$. But then $x \in U \cap V$, and so $U \cap V \neq \emptyset$. This proves that $(\mathbb{R}, F)$ is not a normal space.

Observations from this example:

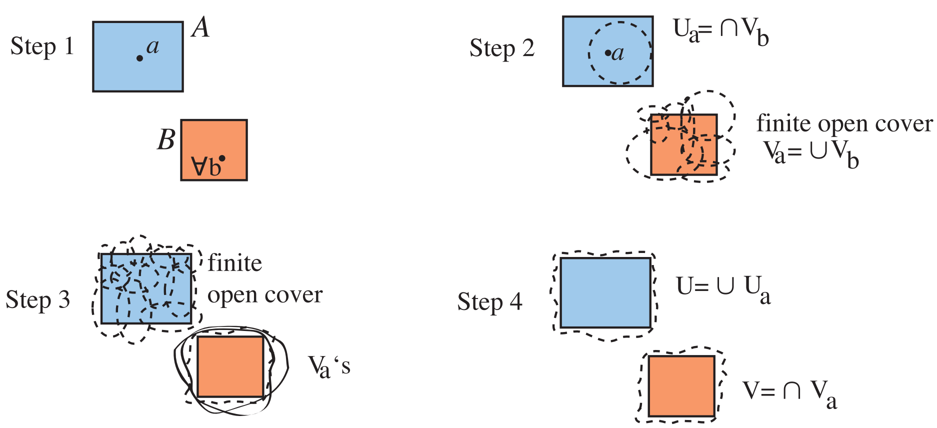
1) A space can be Hausdorff but not normal.

2) Making a topology larger does not change Hausdorff but might change normal because more sets become closed.

Lemma. *If (X, F_x) is Hausdorff and compact, then X is normal.*

Hausdorff ♡ compact forever!!

We begin with the comic book proof.



Proof. Let A and B be disjoint closed subsets of X. Then A and B are compact because X is compact, by the theorem that closed sets in a compact space are compact. Let $a \in A$. For every $b \in B$, there exist open sets U_b and V_b such that $a \in U_b$ and $b \in V_b$, and $U_b \cap V_b = \emptyset$. Then $\{V_b \mid b \in B\}$ is an open cover of B. Since B is compact, we can choose a finite subcover $\{V_{b_1}, \ldots, V_{b_n}\}$. Let $U_a = \cap_{i=1}^n U_{b_i}$. Now $a \in U_a \in F_x$. Let $V_a = \cup_{i=1}^n V_{b_i}$. Then $B \subseteq V_a \in F_x$.

We claim that $U_a \cap V_a = \emptyset$ for all $a \in A$. To see why, note that $(\cap_{i=1}^n U_{b_i}) \cap (\cup_{i=1}^n V_{b_i}) = \emptyset$ because for all i, $U_{b_i} \cap V_{b_i} = \emptyset$.

Now, $\{U_a \mid a \in A\}$ is an open cover of A. Since A is compact, this cover has a finite subcover $\{U_{a_1}, \ldots, U_{a_m}\}$. Now, $V = \cap_{i=1}^m V_{a_i}$ is open and $B \subseteq V$. For all $i = 1, \ldots, m$, $U_{a_i} \cap (\cap_{i=1}^m V_{a_i}) = \emptyset$ because for all $i = 1, \ldots, m$, $U_{a_i} \cap V_{a_i} = \emptyset$.

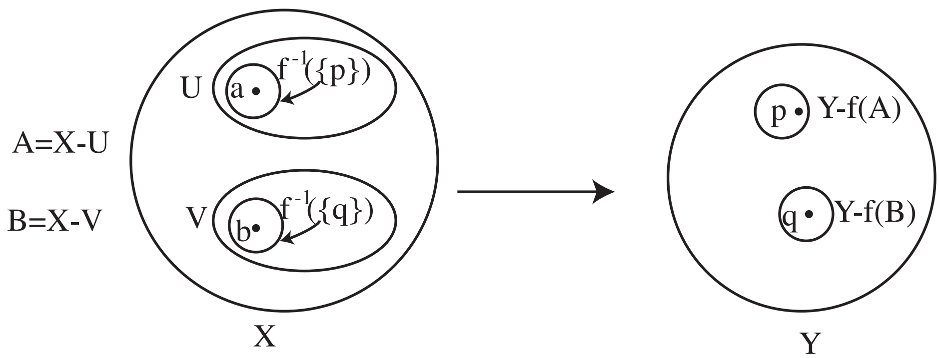
Let $U = \cup_{i=1}^m U_{a_i} \in F_x$. Then $A \subseteq U$ and $U \cap V = \emptyset$ because $U_{a_i} \cap V_{a_i} = \emptyset$ for all i.

Therefore, X is normal. $\square$

Theorem. *Let (X, F_X) be a compact Hausdorff topological space. If (Y, F_Y) is a topological space and $f : X \to Y$ is continuous, onto, and closed, then (Y, F_Y) is compact and Hausdorff.*

Hausdorff ♡ compact forever!!

Proof. I will prove this because the proof is not straight forward. Since f is continuous and onto and X is compact, Y is compact.



To prove Hausdorff: Let $p, q \in Y$ such that $p \neq q$. As f is onto, $\exists\ a, b \in X$ such that $f(a) = p$ and $f(b) = q$. Picking disjoint open sets around a and b won't work because f isn't one to one OR OPEN. As X is Hausdorff, $\{a\}$ and $\{b\}$ are closed. As f is closed, $f(\{a\}) = \{p\}$ and $f(\{b\}) = \{q\}$ are closed. Therefore, as f is continuous, $f^{-1}(\{p\})$ and $f^{-1}(\{q\})$ are closed, and are disjoint. Note we don't need one to one to see that these sets are disjoint.

As X is both compact and Hausdorff, it is also normal by the above Lemma. So $\exists\, U, V \in F_X$ such that $f^{-1}(\{p\}) \subseteq U$, $f^{-1}(\{q\}) \subseteq V$, and $U \cap V = \emptyset$. Note that $f(U)$ and $f(V)$ are not necessarily either disjoint or open. So we consider the complements of U and V. As U and V are open, $A = U^c$ and $B = V^c$ are closed. As f is closed, $f(A)$ and $f(B)$ are also closed, and hence $f(A)^c$ and $f(B)^c$ are open.

We prove as follows that $f(A)^c$ and $f(B)^c$ contain p and q respectively. Suppose that

$p \in f(A)$. Then for some $x \in A$, $f(x) = p$. But then $x \in f^{-1}(\{p\}) \subseteq U = A^c$. So $x \in A \cap A^c$, which is a contradiction. So $p \in f(A)^c$, and by a similar argument, $q \in f(B)^c$.

Now suppose that $y \in f(A)^c \cap f(B)^c$. As f is onto, there exists some $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$, so $z \in A^c = U$. Similarly, as $y \notin f(B)$, $z \in V$. But then $z \in U \cap V = \emptyset$. This is a contradiction, so no such y exists, and hence $f(A)^c \cap f(B)^c = \emptyset$.

So $f(A)^c$ and $f(B)^c$ are disjoint open sets with $p \in f(A)^c$ and $q \in f(B)^c$. As such open sets exist for all $p, q \in Y$, Y is Hausdorff. Therefore Y is compact and Hausdorff, as desired. □

Lemma (Important Lemma about Compact and Hausdorff). *Let $f : (X, F_X) \to (Y, F_Y)$ be continuous. Let X be compact, and Y be Hausdorff. Then f is a homeomorphism if and only if it is a bijection.*

Proof. ($\Rightarrow$) Since f is a homeomorphism, f is a bijection.

($\Leftarrow$) Suppose f is a continuous bijection. We want to show that f is open. Since f is a bijection, this is equivalent to showing that f is closed by HW 3.

Let $A \subseteq X$ be closed. By a theorem in Analysis a closed subset of a compact set is compact. The proof for topological spaces is the same as for metric spaces. So we accept it without proof. Since X is compact, it follows that A is compact. Since f is continuous, $f(A)$ is compact. We previously proved that a compact subset of a Hausdorff space is close. Hence $f(A)$ is closed. Thus, f is closed, and, thus, open.

Thus, f is an *open*, *continuous bijection*. It follows that f is a homeomorphism. Hence, f is a homeomorphism if and only if it is a bijection. □

12. Connected Components

What is the definition of **connected** for metric spaces? What does it mean to say that U and V "form a separation" of X?

Recall the the continuous image of a connected space is connected. The definition and proof of this result are the same for topological spaces. So connectedness is a topological property. Like with compactness, we skip this section because it's too redundant with Analysis. Here are the most important results about connectedness from Analysis:

- A subset $\mathbb{R}$ is connected off it's an interval
- The continuous image of a connected space is connected. So connectedness is a top prop.
- The Flower Lemma (what does it say?)

Feel free to use the results about connectedness in the text together with results from Analysis, but you must state whatever results you're using.

Definition. Let (X, F_X) be a topological space and let $p \in X$. Let $\{C_j \mid j \in J\}$ be the set of all connected subspaces of X containing p. Then $\bigcup_{j\in J} C_j$ is said to be the **connected component**, C_p, of p.

And now, some tiny facts about connected components.

Tiny Fact. Let (X, F_X) be a topological space. Then,

(1) $\forall p \in X$, C_p is connected.

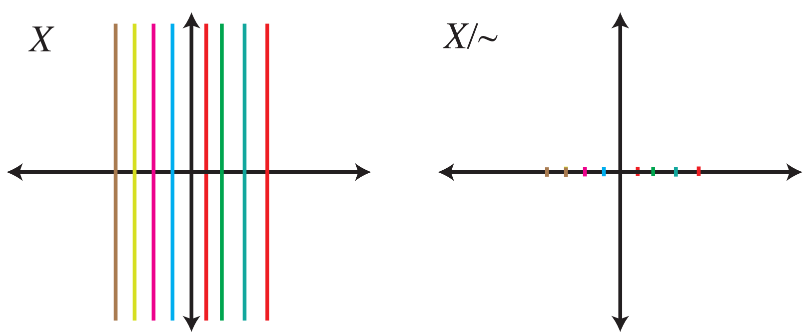
(2) If C_p and C_q are connected components, then either $C_p \cap C_q = \emptyset$ or $C_p = C_q$. (That is, connected components partition a space.)

Proof. 1) $p \in \bigcap_{j\in J} C_j$, so $C_p = \bigcup_{j\in J} C_j$ is connected by the Flower Lemma.

2) Suppose $C_p \cap C_q \neq \emptyset$ and let $x \in C_p \cap C_q$. Then $C_p \cup C_q$ is connected by the Flower Lemma.

$p \in C_p \cup C_q$, so $C_p \cup C_q \in \{C_j \mid j \in J\}$. Also, $C_p \cup C_q \subseteq C_p$, since $C_p = \bigcup_{j\in J} C_j$. So, $C_q \subseteq C_p$. Similarly, $C_p \subseteq C_q$. Thus $C_p = C_q$, and hence X is partitioned into its connected components. □

Example. Let $X = \mathbb{R}^2$ with the dictionary topology. Connected components are vertical lines. (Proof is an exercise.) Define $x \sim y$ $(x, y \in \mathbb{R}^2)$ if x and y are in the same connected component. Then $X/\sim\, = \mathbb{R}$ with the discrete topology.



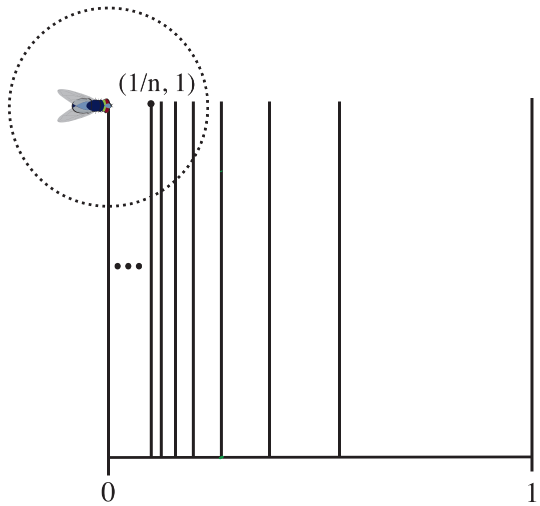
Connectedness is not always intuitive, though. Here's an example showing this.

Define the Comb, the Flea, and the space X as follows. Let $Y_0 = [0,1] \times \{0\}$

and $\forall n \in \mathbb{N}$, $Y_n = \{\frac{1}{n}\} \times [0,1]$.

Comb $= Y = \displaystyle\bigcup_{n=0}^{\infty} Y_n$. Flea$= \{(0,1)\}$.

X =Flea $\cup$ Comb with the subspace topology from $\mathbb{R}^2$. Note, when we studied the comb metric in the beginning of the semester, the topology was not the same as the subspace topology from $\mathbb{R}^2$.



Claim: X is connected. This is surprising because the flea can't get to the comb.

PROOF. First, we see that the Comb is connected as follows. $\forall n \in \{0\} \cup \mathbb{N}$, Y_n is connected because $Y_n \cong [0,1]$. Also, $\forall n \in \mathbb{N}$, $Y_0 \cap Y_n \neq \emptyset$ (because $(\frac{1}{n}, 0)$ is in both), so $Y_0 \cup Y_n$ is connected.

Now $\bigcap_{n \in \mathbb{N}} Y_0 \cup Y_n \neq \emptyset$, so $\bigcup_{n=0}^{\infty} Y_0 \cup Y_n = \bigcup_{n=0}^{\infty} Y_n$ and so the Comb is connected by the Flower Lemma.

Now we want to show that X is connected. Suppose U, V form a separation of X. Since the Comb is connected, WLOG we can assume that Comb $\subseteq U$. Then it must be that Flea $\subseteq V$ because U, V is a separation. Hence V is open in X, so $\exists \varepsilon > 0$ s.t. $B_\varepsilon((0,1); X) \subseteq V$. Let $n \in \mathbb{N}$ s.t. $n > \frac{1}{\varepsilon}$. Then $d((\frac{1}{n}, 1), (0,1)) = \frac{1}{n} < \varepsilon \Rightarrow (\frac{1}{n}, 1) \in B_\varepsilon((0,1); X) \subseteq V$. Observe that since $Y_n = \{\frac{1}{n}\} \times [0,1]$, the point $(\frac{1}{n}, 1) \in Y_n$. Thus, $(\frac{1}{n}, 1) \in V \cap Y_n \subseteq V \cap Y \subseteq V \cap U \Rightarrow V \cap U \neq \emptyset$. But this is a contradiction, since we assumed that U, V was a separation of X. Thus, X is connected. $\square$

Now we have a concept that corresponds more to our intuition about what connected should mean.

13. Path-Connectedness

Definition. Let (X, F_X) be a topological space and let $f : [0,1] \to X$ be continuous. Then we say that f is a **path** from $f(0)$ to $f(1)$ (Note: it is handy to think of $t \in [0,1]$ as time).

Definition. Let (X, F_X) be a topological space. If $\forall p,\ q \in X$, there exists a path in X from p to q then we say that X is **path connected**.

Example. $\mathbb{R}^n$ is path connected:

Let $a,\ b \in \mathbb{R}^n$ be given. Let $f : I \to \mathbb{R}^n$ be $f(t) = (1-t)a + bt$. Then $f(0) = a$, and $f(1) = b$. f can be shown to be continuous.

Some remarks:

(1) The above path is important and will arise frequently in the rest of the course.

(2) Connected is a negative definition and path connected is a positive definitionTo prove connectedness, we are trying to show that a separation *doesn't* exist, so it is easiest to do connectedness proofs by contradiction or use the Flower Lemma.

(3) To prove path connectedness, we are trying to show that a path exists. So path connectedness proofs are more easily done constructively.

(4) In general, it is easier to prove that a space is disconnected than to prove that it is not path connected.

(5) In general, it is easier to prove that a space is path connected than to prove that it is connected.

Theorem. *If (X, F_X) is path connected, then (X, F_X) is connected.*

Proof. Suppose there exists a separation U, V of X. Since U, V is a separation of X, $U \neq \emptyset$, $V \neq \emptyset$. Let $p \in U$, $q \in V$ be given. Since X is path connected, $\exists$ a path f from p to q. Since paths are continuous by definition, f is continuous, so $f^{-1}(U)$, $f^{-1}(V)$ are open in $[0,1]$.

Claim 1: $f^{-1}(U) \cup f^{-1}(V) = [0,1]$.
Proof of Claim 1: Let $x \in [0,1]$ be given. Then $f(x) \in X = U \cup V \Rightarrow f(x) \in U$ or $f(x) \in V \Rightarrow x \in f^{-1}(U)$ or $x \in f^{-1}(V) \Rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$. Thus, $[0,1] \subseteq f^{-1}(U) \cup f^{-1}(V) \Rightarrow [0,1] = f^{-1}(U) \cup f^{-1}(V)$ (since $f^{-1}(U)$, $f^{-1}(V) \subseteq [0,1]$).

Claim 2: $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.
proof of Claim 2: Suppose $\exists x \in f^{-1}(U) \cap f^{-1}(V)$. Then $x \in f^{-1}(U) \Rightarrow f(x) \in U$, and $x \in f^{-1}(V) \Rightarrow f(x) \in V$, so $f(x) \in U \cap V$, which is impossible since we are assuming that U and V are a separation of X. Thus, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Observe that since f is a path from p to q, $f(0) = p$ and $f(1) = q$, so $0 \in f^{-1}(U)$, $1 \in f^{-1}(V)$ so $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty and proper. Thus, since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$

and $f^{-1}(U) \cup f^{-1}(V) = [0,1]$, $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of $[0,1]$. But this is a contradiction, since we know from Math 131 that $[0,1]$ is connected. Thus, (X, F_X) must be connected. □

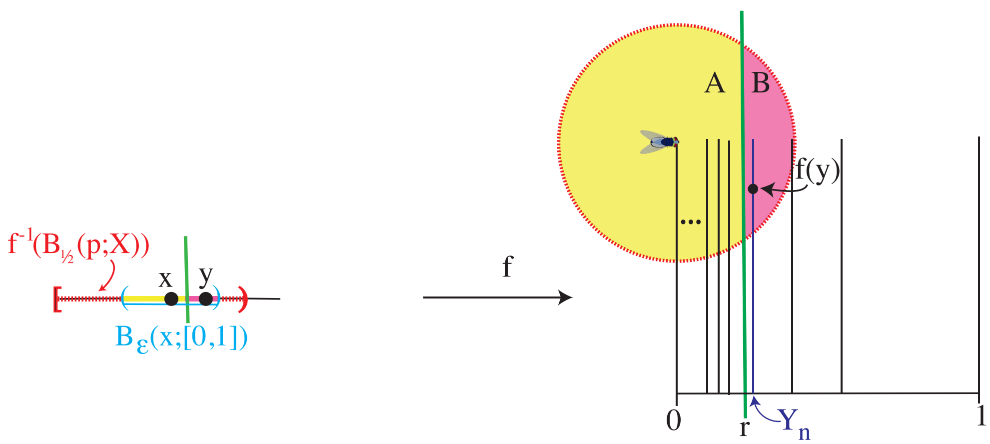
Recall that the flea and comb space is connected.

Theorem. *Let X denote the flea and comb space. Then X is not path connected.*

Proof. We want to show that $\nexists$ a path from the flea to the comb.

Suppose $\exists$ a path f from the flea to a point on the comb. Let $p =$ flea. Then $f^{-1}(\{p\})$ is not empty because $f(0) = p$ by the definition of f. Similarly, by the definition of f, $f(1) \neq p$, so $f^{-1}(\{p\})$ is a non-empty proper subset of I. Observe that since $\{p\}$ is closed in X and f is continuous, $f^{-1}(\{p\})$ is closed in I.

We'd like to show that $f^{-1}(\{p\})$ is open in I, since then it would be a non-empty proper clopen subset of I, giving us a contradiction. Let $x \in f^{-1}(\{p\})$. We will prove that there is an open interval U such that $x \in U \subseteq f^{-1}(\{p\})$.



Outline of rest of proof:

(1) $f^{-1}(B_{\frac{1}{2}}(p;X))$ is open and contains x.

(2) Hence it contains an open interval $B_\varepsilon(x;[0,1])$. WTS $B_\varepsilon(x;[0,1]) \subseteq f^{-1}(\{p\})$.

(3) Suppose some point $y \in B_\varepsilon(x;[0,1])$ is not in $f^{-1}(\{p\})$.

(4) Then $f(y) \in Y_n$ for some n.

(5) Now separate $f(B_\varepsilon(x;[0,1]))$ by a vertical line at a irrational x value so that one open set contains p and the other contains $f(y)$.

(6) But $f(B_\varepsilon(x;[0,1]))$ is connected since f is continuous and intervals are connected.

(7) Thus $B_\varepsilon(x;[0,1]) \subseteq f^{-1}(p)$.

Observe that $B_{\frac{1}{2}}(p;X)$ is open in X. Since f is a path, f is continuous, so $f^{-1}(B_{\frac{1}{2}}(p;X))$ is open in $[0,1]$. Since $x \in f^{-1}(\{p\})$, $f(x) = p \in B_{\frac{1}{2}}(p;X)$. Thus $x \in f^{-1}(B_{\frac{1}{2}}(p;X))$. Now, since $f^{-1}(B_{\frac{1}{2}}(p;X))$ is open in $[0,1]$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x;[0,1]) \subseteq f^{-1}(B_{\frac{1}{2}}(p;X))$.

Let $y \in B_\varepsilon(x;[0,1])$. We will show that $f(y) = p$, and hence $B_\varepsilon(x;[0,1])$ will be the open interval U such that $x \in U \subseteq f^{-1}(\{p\})$. This will prove that $f^{-1}(\{p\})$ is open.

Suppose $f(y) \neq p$. Since $y \in B_\varepsilon(x;[0,1])$, $y \in f^{-1}(B_{\frac{1}{2}}(p;X))$, so $f(y) \in B_{\frac{1}{2}}(p;X)$ (so $d(f(y),p) < \frac{1}{2}$). For each $q \in Y_0 = [0,1] \times \{0\}$, we know that $d(p,q) \geq 1$. Hence $q \notin B_{\frac{1}{2}}(p;X)$. Thus, $f(y) \notin Y_0$. Thus since $f(y) \neq p$, $\exists n \in \mathbb{N}$ s.t. $f(y) \in Y_n$.

Let $r \in \mathbb{R} - \mathbb{Q}$ such that $0 < r < \frac{1}{n}$. Let $A = \{(s,t) \in f(B_\varepsilon(x;[0,1]))\,|s < r\}$ and $B = \{(s,t) \in f(B_\varepsilon(x;[0,1]))\,|s > r\}$.

Claim: A and B is a separation of $f(B_\varepsilon(x;[0,1]))$.

Note that $A \cap B = \emptyset$ by definition. We know that $f(y) \in B$, and $p \in A$, so neither set is empty. Next, we want to show that $A \cup B = f(B_\varepsilon(y;[0,1]))$. Certainly $A \cup B \subseteq f(B_\varepsilon(x;[0,1]))$. Now let $(s,t) \in f(B_\varepsilon(x;[0,1]))$. Since $f(B_\varepsilon(x;[0,1])) \subseteq B_{\frac{1}{2}}(p;X)$, $(s,t) \notin Y_0$. Thus $s = \frac{1}{m}$ for some $m \in \mathbb{N}$. Then $s \neq r$, so $(s,t) \in A \cup B$ and $A \cup B = f(B_\varepsilon(x;[0,1]))$.

Next we want to show that A is open in $f(B_\varepsilon(x;[0,1]))$. We know that $\{(s,t)|s < r\}$ is open in $\mathbb{R}^2$. Hence, since X has the subspace topology $A = \{(s,t)|s < r\} \cap f(B_\varepsilon(x;[0,1]))$ is open in $f(B_\varepsilon(x;[0,1]))$. Similarly, B is open in $f(B_\varepsilon(x;[0,1]))$.

We have shown that A and B is a separation of $f(B_\varepsilon(x;[0,1]))$. $\surd$

This is a contradiction, since $B_\varepsilon(x;[0,1])$ is connected and f is continuous. We conclude that $f(y) = p$, for all $y \in B_\varepsilon(x;[0,1])$.

Therefore $B_\varepsilon(x;[0,1]) \subseteq f^{-1}(\{p\})$, so $f^{-1}(\{p\})$ is open. So $f^{-1}(\{p\})$ is a clopen, non-empty, proper subset of $[0,1]$. This is a contradiction, so we conclude that there does not exist a path in X from p to a point in the comb. $\square$

The point of this whole example is to show that path connected is stronger than connected. We knew already that path connected implies connected, and now we see that connected doesn't necessarily imply path connected.

Lemma. *The continuous image of a path connected set is path connected.*

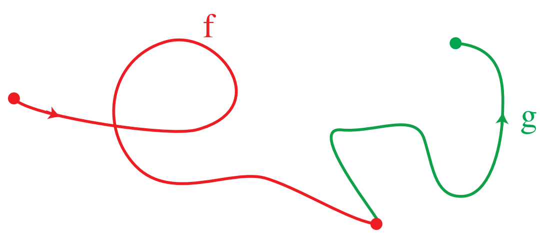
Proof. Let X be path connected, $g : X \to Y$ be continuous, let p and q be points in $g(X)$. Then there are points r and s in X such that $g(r) = p$ and $g(s) = q$. Since X is path connected, there is a path $f : [0,1] \to X$ such that $f(0) = r$ and $f(1) = s$. Now $g \circ f : [0,1] \to Y$ is a path from $g(f(0)) = p$ to $g(f(1)) = q$. Hence $f(X)$ is path connected. $\square$

13.1. Combining paths.

DEFINITION. Let f and g be paths in a topological space (X, F_X) such that $f(1) = g(0)$. Then we define $f * g : I \to X$ by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Intuitively what's happening is that we're connecting two paths while speeding things up, creating a new single path parametrized from 0 to 1.



SMALL FACT. $f * g$ is a path from $f(0)$ to $g(1)$.

PROOF. By the Pasting Lemma[2], since $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed subsets under $[0, 1]$, and $f(2(\frac{1}{2})) = f(1) = g(0) = g(2(\frac{1}{2}) - 1)$, $f * g$ is continuous. So $f * g$ is a path. Since $(f * g)(0) = f(0)$ and $(f * g)(1) = g(1)$, then $f * g$ is a path from $f(0)$ to $f(1)$. □

This definition also allows us to create an analogue to the flower lemma.

THEOREM (Flower Lemma for Path Connected Sets). *Let $X = \bigcup_{i \in J} Y_i$ such that $\forall i \in J$, Y_i is path connected, and $Y_i \cap Y_{i_0} \neq \emptyset$. Then X is path connected.*

PROOF. Let $a, b \in X$. If $\exists j \in J$ such that $a, b \in Y_j$, then there exists a path from a to b in $Y_j \subseteq X$. So without loss of generality, suppose that $a \in Y_j, b \in Y_k$, and $k \neq j$. Let $x \in Y_{i_0} \cap Y_j$ and $y \in Y_{i_0} \cap Y_k$. Then there exists a path f from a to x in Y_j, a path h from x to y in Y_{i_0}, and a path g from y to b in Y_k. So $(f * h) * g$ is a path in X from a to b, and we're done. (Note that we need parentheses in $(f * h) * g$, since otherwise it's not defined.) □

COROLLARY. *The product of path connected spaces is path connected.*

PROOF. Let $x \in X$. Then $Y_0 = \{x\} \times Y \cong Y$ is path connected. Also, for each $j \in Y$, we know that $Y_j = X \times \{j\} \cong X$ is path connected. Since $Y_j \cap Y_o \neq \emptyset$ for each $j \in Y$, by the Flower Lemma for Path Connected Sets, $X \times Y$ is path connected. □

[2] HW3 #1, which states that if we have two continuous functions with closed sets as domains, and they agree over the intersection of these domains, then the combined function is continuous

Definition. Let (X, F_X) be a topological space, and $p \in X$. Let $\{C_j | j \in J\}$ be the set of all path connected subspaces of X containing p. Then $\bigcup_{j \in J} C_j$ is said to be the **path connected component** C_p.

Some tiny facts about path connected components:

Let (X, F_X) be a topological space. Then

(1) $\forall p \in X, C_p$ is path connected

(2) If C_p, C_q are path connected components, then either $C_p \cap C_q = \emptyset$, or $C_p = C_q$. In other words, path connected components partition the set.

We'll omit this proof, as it is identical to the one we did for connected components.

14. Homotopies

Now we start the algebraic/geometric part of the course, which you will see has a substantially different flavor from the previous parts of the course

Definition. Let f be a path in (X, F_X), and define $\bar{f} : I \to X$ by $\bar{f}(t) = f(1-t)$.

A few remarks:

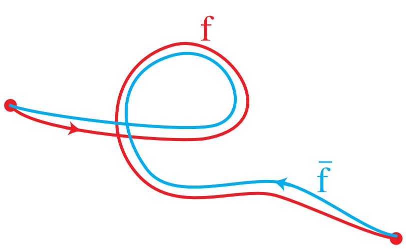
(1) $\bar{f}$ is a path, because it is a composition of continuous functions.

(2) $\bar{f}$ is a path from $f(1)$ to $f(0)$.

(3) $*$ is a "multiplication" of paths, and $\bar{f}$ seems like an inverse path. But $f * \bar{f} \neq$ the "identity". Even so, this gives us the idea of forming a group with paths as elements. But we aren't quite ready to do this.

Definition. Let (X, F_X) be a topological space, and $a \in X$. We define $e_a : I \to X$ by $e_a(t) = a$ for all $t \in I$.

e_a is the constant path at a. We might want $f * \bar{f} = e_a$. But this isn't true.



Also $e_a * f \neq f$ since $e_a * f$ hangs out at a for half a minute, then does f at twice the usual speed.

So we can't just make a group out of paths. Instead we consider deformations of paths. Deformations seem good because we have the intuition that if one space can be deformed to another then they should be equivalent. We need to formalize what we mean by "deforming" one path to another. Intuitively it should mean that there is a continuous way to change one path into another.

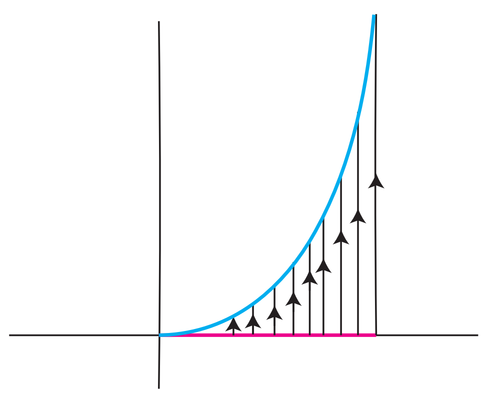
We start with a general notion of deforming continuous functions. We will return to paths later.

Definition. Let (X, F_X) and (Y, F_Y) be top spaces, and let $f_0 : X \to Y$ and $f_1 : X \to Y$ be continuous. Then we say that f_0 is **homotopic** to f_1 if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$, and $F(x, 1) = f_1(x)$. We say that F is a *homotopy* from f_0 to f_1, and we write $f_0 \simeq f_1$.

This is the most important concept for the rest of the course. The idea is that we can deform one function to the other over time. Note that x is the variable of the function and t is the time variable for the homotopy. From now on when we have paths, we'll use the variable s for the path.

It is useful to recall that if a and b are points in $\mathbb{R}^n$, then the straight line path from a to b is given by $f(s) = sb + (1 - s)a$.

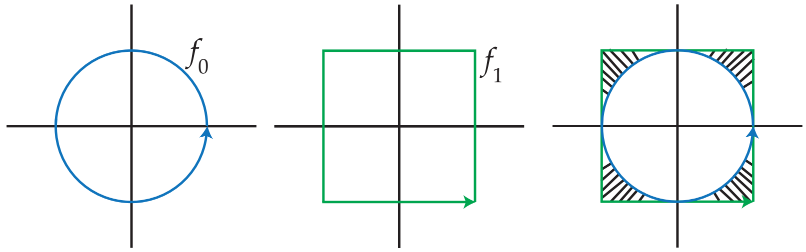
Example. Let $X = I$ and $Y = \mathbb{R}^2$. Define $f_0 : I \to \mathbb{R}^2$ by $f_0(s) = (s, 0)$, $f_1 : I \to \mathbb{R}^2$ by $f_1(s) = (s, s^2)$, and $F : I \times I \to \mathbb{R}^2$ by $F(s, t) = t(s, s^2) + (1 - t)(s, 0)$.



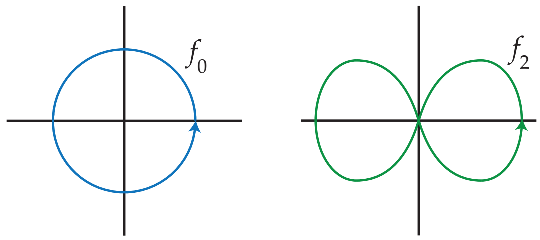
Observe that F is continuous, since it's a composition of continuous functions. Note also that $F(s, 0) = (s, 0) = f_0(s)$ and $F(s, 1) = (s, s^2) = f_1(s)$. Thus F is a homotopy from f_0 to f_1.

The homotopy in the above example is called the *straight line homotopy*. We can define a straight line homotopy for any pair f_0 and f_1 of continuous functions from a space X to $\mathbb{R}^n$, by $F(x, t) = tf_1(x) + (1 - t)f_0(x)$. However, the straight line homotopy only makes sense in $\mathbb{R}^n$ or a convex subspace.

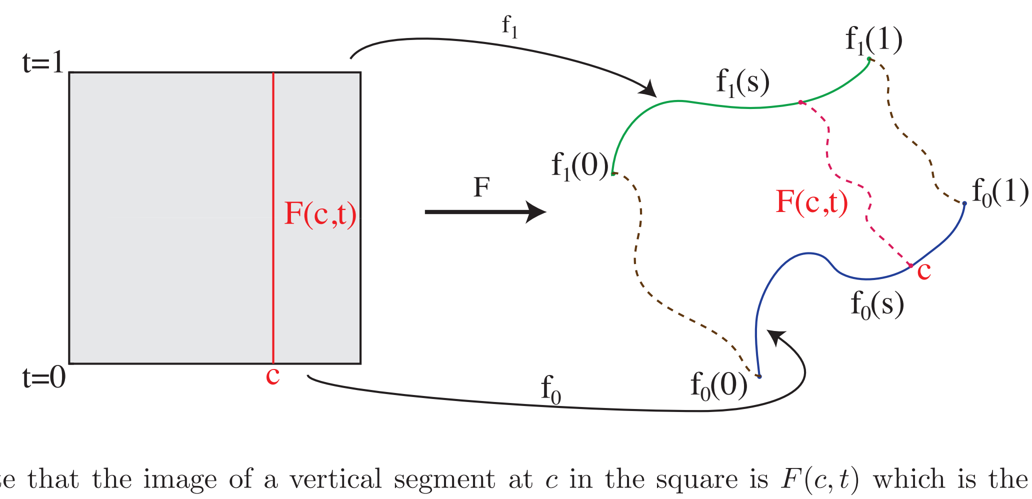
To illustrate homotopies, consider a couple of pictures of functions from a circle to $\mathbb{R}^2$. We can use the straight line homotopy to take one to the other, so f_0 and f_1 are homotopic.



Note that the images of homotopic functions need not be homeomorphic. For example, the following functions are homotopic by the straight line homotopy.



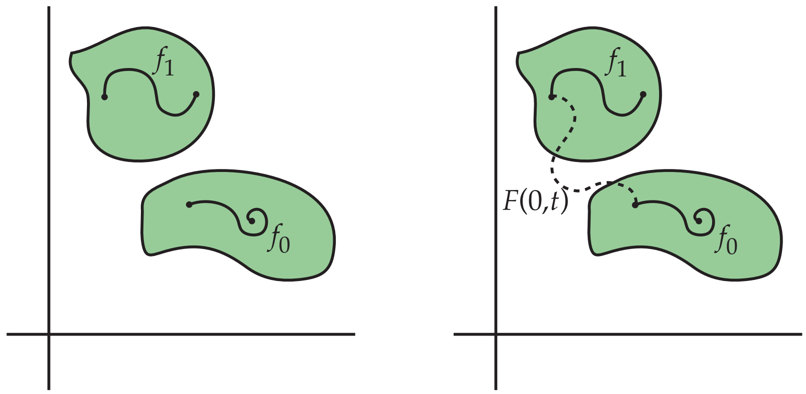
14.1. Drawing Homotopies. We find it useful to express a homotopy with a picture. We do this by drawing X as an interval so that the domain of F is drawn as a square. The X axis is horizontal, and the I axis is vertical. Then the bottom of the square represents $X \times \{0\}$ hence when F is restricted to the bottom of the square we get f_0. Similarly, when F is restricted to the top of the square we get f_1. Thus we have the following diagram representing the homotopy.



Note that the image of a vertical segment at c in the square is $F(c,t)$ which is the path taken from $f_0(c)$ to $f_1(c)$.

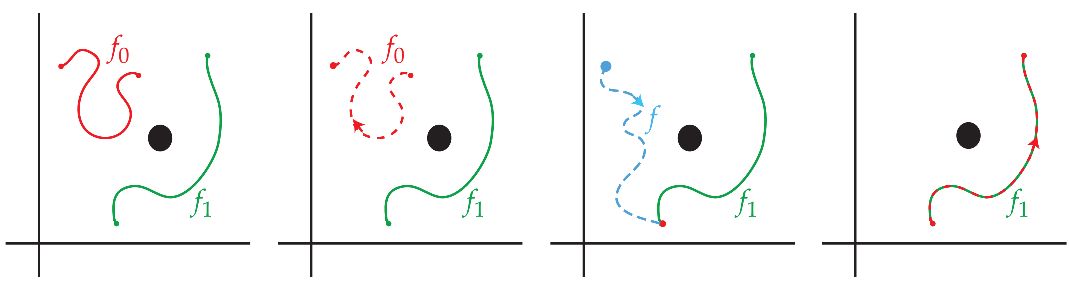
A few remarks.

(1) Suppose Y is not path connected and $f_0(x)$ and $f_1(x)$ are in different path components. Then there does not exist a homotopy from f_0 to f_1. Why?



If f_0 and f_1 were homotopic, then $F(0,t)$ would be a path from $f_0(0)$ to $f_1(0)$. But f_0 and f_1 are in different path components.

(2) If Y is path connected and f_0 and f_1 are paths in Y, then $f_0 \simeq f_1$. That is any pair of paths is homotopic. Homotop (the verb!) f_0 to its initial point, move it to the initial point of f_1 and then stretch it back out into f_1.)



More formally, we prove this as follows.

Proof: Define $G : I \times I \to Y$ by $G(s,t) = f_0(t \cdot 0 + (1-t)s)$. Observe that G is continuous, and $G(s,0) = f_0(s)$ and $G(s,1) = f_0(0)$. So G homotops f_0 to its initial point.

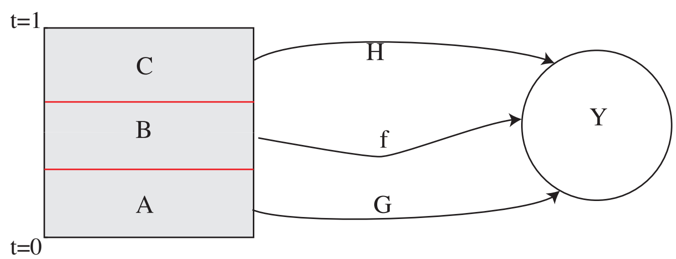
Let $f : I \to Y$ be a path from $f_0(0)$ to $f_1(0)$.

Now define $H : I \times I \to Y$ by $H(s,t) = f_1(ts + (1-t) \cdot 0)$. Observe that H is continuous and $H(s,0) = f_1(0)$ and $H(s,1) = f_1(s)$. So H homotops $f_1(0)$ to f_1.

Finally, define $F : I \times I \to Y$ by

$$F(s,t) = \begin{cases} G(s,3t) & t \in [0, \frac{1}{3}] \\ f(3t-1) & t \in [\frac{1}{3}, \frac{2}{3}] \\ H(s, 3t-2) & t \in [\frac{2}{3}, 1] \end{cases}$$

This function does G at triple speed starting at $t = 0$ (so no shift is necessary), then does f at triple speed starting at $t = \frac{1}{3}$ (so we need to shift by 1), finally it does H at triple speed starting at $\frac{2}{3}$ (so we need to shift by 2). We illustrate the homotopy as follows. Note that since we are speeding s up by 3, there are three rectangles stacked vertically.



Do the rest of this in the round

F is continuous: Let $A = I \times [0, \frac{1}{3}]$ and $B = I \times [\frac{1}{3}, \frac{2}{3}]$ and $C = [\frac{2}{3}, 1]$. All are closed in $I \times I$. We need to check that G and f agree on $A \cap B = I \times \frac{1}{3}$ and f and H agree on $B \cap C = I \times \frac{2}{3}$. Then we use the Pasting Lemma to conclude that F continuous.

F is a homotopy between f_0 and f_1:

$$F(s,0) = G(s,0) = f_0(s) \qquad \text{and} \qquad F(s,1) = H(s,1) = f_1(s)$$

Thus F is indeed an homotopy, and f_0 is homotopic to f_1.

What path does $s = 0$ take during this homotopy? What path does $s = 1$ take?

15. Homotopy Equivalence

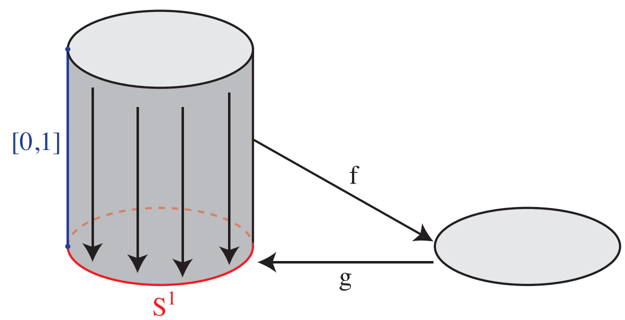
Definition. Let (X, F_x), (Y, F_y) be topological spaces. We say X and Y are **homotopy equivalent** if there exist continuous f, g where $f : X \to Y$, $g : Y \to X$, such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

Example. If f is a homeomorphism, then X and Y are homotopy equivalent (using the functions f, f^{-1}).

As the following example illustrates, homotopy equivalent is weaker than homeomorphic.

Example. Let $X = S^1 \times I$ and $Y = S^1$. Then $S^1 \times I \ncong S^1$, as removing 2 points disconnects S^1, but does *not* disconnect $S^1 \times I$.

Let $f : S^1 \times I \to S^1$ by $f(x, y) = x$, and let $g : S^1 \to S^1 \times I$ by $g(x) = (x, 0)$. These are continuous but we don't prove it.



Then $(f \circ g)(x) = f(x, 0) = x$, so $f \circ g = id_Y$.

It remains to show that $(g \circ f)(x) \simeq id_X$.

Define $F : (S^1 \times I) \times I \to S^1 \times I$ by

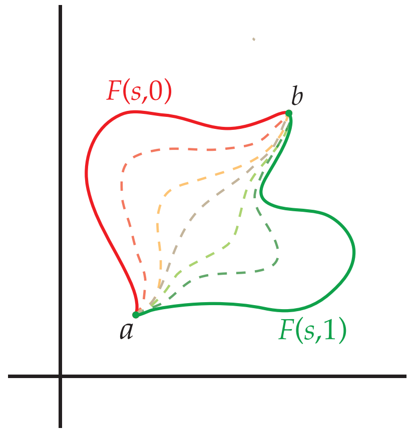
$$F((x, y), t) = (x, yt)$$

As F is the composition of continuous functions it is continuous.

$F((x, y), 0) = (x, 0) = g \circ f(x, y)$, and $F((x, y), 1) = (x, y) = id_X(x, y)$. Thus F is a homotopy. Hence X and Y are homotopy equivalent.

15.1. Path Homotopies. Now we return to our study of paths. Recall that in a path connected space all paths are homotopic. So we need a stronger type of homotopy if we want to get more interesting results. The following type of homotopy does not all us to to move the endpoints of the paths.

Definition. Let f_0 and f_1 be paths in (X, F_X) from a to b. We say f_0 is *path-homotopic* if there exists a homotopy F from f_0 to f_1 s.t. $\forall t \in I$, $F(0, t) = a$ and $F(1, t) = b$. We say F is a **path homotopy** and write $f_0 \sim f_1$.



Note that there can be no path homotopy between paths which don't have the same endpoints.

Example. Let X be a convex region of $\mathbb{R}^n$, let a, $b \in X$, and let f_0 and f_1 be paths in X from a to b.

Claim: $f_0 \sim f_1$.

Proof. Let $F(s,t) = (1-t)f_0(s) + tf_1(s)$. Then F is the straight line homotopy from f_0 to f_1. Now let $t \in I$ be given. Observe that $F(0,t) = (1-t)f_0(0) + tf_1(0) = a$ because $f_1(0) = f_0(0) = a$. Similarly, $F(1,t) = f_0(1) = b$. Thus $\forall t \in I$, $F(0,t) = a$ and $F(1,t) = b$, so F is a path homotopy, and thus $f_0 \sim f_1$. $\square$

Example. Let $X \cong D^2$, let $a, b \in X$, and let f_0 and f_1 be paths in X from a to b. Then $f_0 \sim f_1$.

Proof. Let $g : X \to D^2$ be a homeomorphism. Let $F : (I \times I) \to D^2$ be the straight line homotopy in D^2 from $g \circ f_0$ to $g \circ f_1$ (Note: D^2 is a convex region of $\mathbb{R}^2$, so by the last example we can use the straight line homotopy here).

First, we need to show that $g^{-1} \circ F$ is continuous. Note that F is continuous since F is a homotopy. Note also that since g is a homeomorphism, g^{-1} is continuous. Thus $g^{-1} \circ F$ is the composition of continuous functions, and hence $g^{-1} \circ F$ is continuous.

Now we need to show that $g^{-1} \circ F$ is a homotopy from f_0 to f_1.

First, observe that $\forall s \in I$, $(g^{-1} \circ F)(s,0) = g^{-1}((1-0)g(f_0(s)) + (0)g(f_1(s))) = g^{-1}(g(f_0(s))) = f_0(s)$ since g is a bijection, and similarly $(g^{-1} \circ F)(s,1) = f_1(s)$. Thus, since $g^{-1} \circ F$ is continuous, $g^{-1} \circ F$ is a homotopy from f_0 to f_1.

Lastly, we need to show that $g^{-1} \circ F$ is a *path* homotopy.

Note that $\forall t \in I$, $(g^{-1} \circ F)(0,t) = g^{-1}((1-t)g(f_0(0)) + tg(f_1(0))) = g^{-1}((1-t)g(a) + tg(a)) = g^{-1}(g(a)) = a$ since g is a bijection. Similarly, $\forall t \in I$, $(g^{-1} \circ F)(1,t) = b$. Thus, since $g^{-1} \circ F$ is a homotopy from f_0 to f_1, $g^{-1} \circ F$ is a path homotopy, and thus $f_0 \sim f_1$. $\square$

Theorem. *Let (X, F_X) be a topological space and let a, $b \in X$ be given. Then $\sim$ is an equivalence relation on paths in X from a to b.*

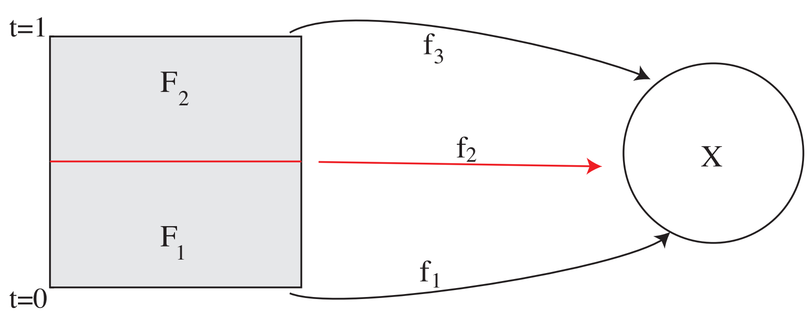
Proof. In order to show $\sim$ is an equivalence relation, we need to show that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: If f is a path in X from a to b, let $F : (I \times I) \to X$ be given by $F(s,t) = f(s)$. Note that F is a homotopy from f to f since, $\forall s \in I$, $F(s,0) = f(s)$ and $F(s,1) = f(s)$ and F is continuous since f is continuous. Observe that $\forall t \in I$, $F(0,t) = f(0) = a$ and $F(1,t) = f(1) = b$, so F is a path homotopy and $f \sim f$. Thus, $\sim$ is reflexive.

Symmetric: Suppose that $f_1 \sim f_2$. Then there exists a path homotopy F from f_1 to f_2 . Define $F' : (I \times I) \to X$ given by $F'(s,t) = F(s, 1-t)$, $\forall (s,t) \in (I \times I)$. It is not hard to check that F' is a path homotopy, and thus $f_2 \sim f_1$, so $\sim$ is symmetric.

Transitive: Suppose that f_1, f_2, and f_3 are paths in X from a to b s.t. $f_1 \sim f_2$ and $f_2 \sim f_3$. Since $f_1 \sim f_2$, there exists a path homotopy F_1 from f_1 to f_2, and since $f_2 \sim f_3$, there exists a path homotopy F_2 from f_2 to f_3. Define $F_3 : (I \times I) \to X$ by

$$F_3(s,t) = \begin{cases} F_1(s, 2t) & t \in [0, \frac{1}{2}] \\ F_2(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$



Now we must show that F_3 is a path homotopy from f_1 to f_3. Observe that $A = I \times [0, \frac{1}{2}]$ and $B = I \times [\frac{1}{2}, 1]$ are closed in $I \times I$, and F_1 and F_2 are continuous, so if $F_1(s,t) = F_2(s,t)$ $\forall (s,t) \in A \cap B$, then by the pasting lemma F_3 is continuous. Since $A \cap B = I \times \{\frac{1}{2}\}$, and, $\forall s \in I$, $F_1(s, 2(\frac{1}{2})) = F_1(s,1) = f_2(s)$ and $F_2(s, 2(\frac{1}{2}) - 1) = F_2(s,0) = f_2(s)$, F_3 is continuous by the pasting lemma. Now, observe that $\forall s \in I$, $F_3(s,0) = F_1(s, 2(0)) = F_1(s,0) = f_1(s)$ and $F_3(s,1) = F_2(s, 2(1)-1) = F_2(s,1) = f_3(s)$, so F_3 is a homotopy from

f_1 to f_3. Now, observe that $\forall t \in I$,

$$F_3(0,t) = \begin{cases} F_1(0,2t) & \text{if } t \in [0,\frac{1}{2}] \\ F_2(0,2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases}$$
$$= \begin{cases} a & t \in [0,\frac{1}{2}] \\ a & t \in [\frac{1}{2},1] \end{cases}$$

(since F_1 and F_2 are path homotopies), and thus $F_3(0,t) = a$, $\forall t \in I$. Similarly, $\forall t \in I$, $F_3(1,t) = b$. Thus, F_3 is a path homotopy from f_1 to f_3, so $f_1 \sim f_3$, and thus $\sim$ is transitive.

Thus, $\sim$ is an equivalence relation of paths in X from a to b. $\square$

Note that the same proof (using only the parts related to continuity and homotopy) works for $\simeq$.

Definition. Let (X, F_X) be a topological space and let $a, b \in X$ be given. For each path f from a to b in X, define $[f]$ to be the **path homotopy class of f**.

Definition. Let f be a path in X from a to b and g be a path in X from b to c. Define an **invisible symbol** by $[f][g] = [f * g]$.

Some remarks:

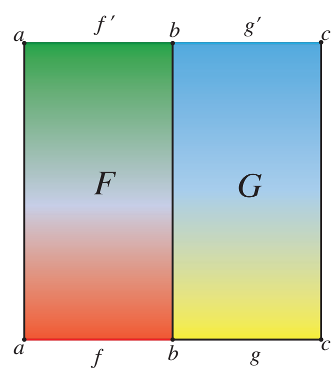
(1) $[f]$, $[g]$, and $[f * g]$ are not elements of the same quotient 'world' unless $a = b = c$.

(2) We have to prove that invisible multiplication is well-defined, i.e. if $f \sim f'$ and $g \sim g'$, then we want $[f * g] = [f' * g']$ because $[f] = [f']$ and $[g] = [g']$.

Lemma (Important). *Let (X, F_x) be a topological space. Let f, f' be paths in X from a to b and let g, g' be paths in X from b to c. If $f \sim f'$ and $g \sim g'$, then $f * g \sim f' * g'$ and hence $[f][g] = [f'][g']$.*

Proof. Let F be the path homotopy from f to f', and let G be the path homotopy from g to g'.

As we can see from the following diagram, we just need to speed things up in the s variable.



Define $H : I \times I \to X$ by:

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,1/2] \\ G((2s-1),t) & s \in [1/2,1] \end{cases}$$

We leave it as an exercise to check that H is a homotopy from $f * g$ to $f' * g'$. $\square$

16. Loops and the Fundamental Group

We have shown that the product of two path homotopy classes is well defined. For the purposes of defining a group of path homotopy classes, we would like all paths to have the same starting and ending point so that they can be combined. This simplification motivates the two definitions which follow:

Definition. Let f be a path in X such that $f(0) = f(1) = x_0 \in X$. Then f is said to be a **loop** in X based at x_0.

Note that if f, g are loops in X based at some point $x_0 \in X$, then their product $f * g$ is also a loop based at x_0. In particular, we then have that $[f], [g]$ and $[f][g] = [f * g]$ are all path homotopy classes of loops based at x_0.

Definition. Let (X, F_x) be a topological space, and let $x_0 \in X$. Define $\pi_1(X, x_0)$ as the set of path homotopy classes of loops based at x_0 endowed with the operation $[f][g] = [f*g]$. We call $\pi_1(X, x_0)$ the **fundamental group of X based at** x_0.

16.1. The Fundamental Group is a Group.

Our ultimate goal is to harness the power of group theory from abstract algebra to study topological spaces. But first we must prove that $\pi_1(X, x_0)$ is actually a group. In other words, if (X, F_x) is a topological space with $x_0 \in X$, we must prove the following:

(1) $\pi_1(X, x_0)$ is closed under the invisible operation. In other words, for $[f], [g] \in \pi_1(X, x_0)$, $[f][g] \in \pi_1(X, x_0)$.

(2) The invisible operation is associative. In other words, given $[f], [g], [h] \in \pi_1(X, x_0)$:
$$([f][g])[h] = [f]([g][h])$$

(3) $\pi_1(X, x_0)$ contains an identity element. In other words, there exists $[e_{x_0}] \in \pi_1(X, x_0)$ such that for all $[f] \in \pi_1(X, x_0)$:
$$[e_{x_0}][f] = [f][e_{x_0}] = [f]$$

(4) Every element of $\pi_1(X, x_0)$ has an inverse. In other words, given $f \in \pi_1(X, x_0)$, there exists $g \in \pi_1(X, x_0)$ such that:
$$[f][g] = [g][f] = [e_{x_0}]$$

If $\pi_1(X, x_0)$ satisfies all of these requirements, $\pi_1(X, x_0)$ is a group.

Lemma (Closure). *Let* (X, F_x) *be a topological space, and let* $x_0 \in X$. *Let* $[f], [g] \in \pi_1(X, x_0)$. *Then:*

$$[f][g] \in \pi_1(X, x_0)$$

Proof. We know by a previous result that $f * g$ is a path in X from x_0 to x_0 since x_0 is both the starting point of $[f]$ and the endpoint of $[g]$. Consequently, $f * g$ is a loop in X based at x_0, so $[f * g] \in \pi_1(X, x_0)$. □

Next, we will show that products of path homotopy classes of loops based at a point are associative.

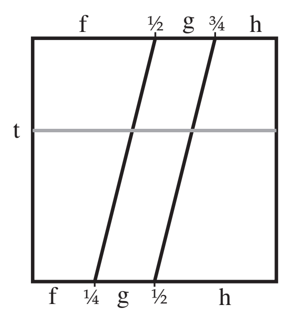
Lemma (Associativity). *Let* (X, F_x) *be a topological space, and let* f, g, *and* h *be paths in* X *such that* $f(1) = g(0)$ *and* $g(1) = h(0)$. *Then:*

$$([f][g])[h] = [f]([g][h])$$

Proof. Before actually proving the result, we write the definitions of $(f * g) * h$ and $f * (g * h)$:

$$(f * g) * h = \begin{cases} f(4s) & s \in [0, \frac{1}{4}] \\ g(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ h(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases} \qquad f * (g * h) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(4s - 2) & s \in [\frac{1}{2}, \frac{3}{4}] \\ h(4s - 3) & s \in [\frac{3}{4}, 1] \end{cases}$$

We need to construct a homotopy from $(f * g) * h$ to $f * (h * g)$. We get the main idea from looking at the picture.



In this and future situations, we use the following

Algorithm to create homotopies:

(1) At an arbitrary time t, define the s intervals where you do each function.

(2) Determine the length of the intervals.

(3) Find the speed of each function on that interval by taking the reciprocal of the length.

(4) Determine the shift so that it starts at the right time.

Based on our formulas for $(f * g) * h$ and $f * (g * h)$, we define the function $F : I \times I \to X$ by:

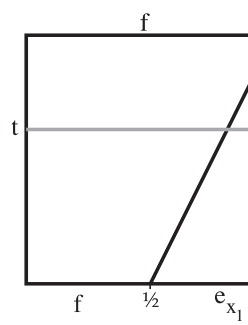
$$F(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & s \in \left[0, \frac{1+t}{4}\right] \\ g(4s-1-t) & s \in \left[\frac{1+t}{4}, \frac{2+t}{4}\right] \\ h\left(\frac{4s}{2-t} - \frac{2+t}{2-t}\right) & s \in \left[\frac{2+t}{4}, 1\right] \end{cases}$$

We leave it as an exercise to check that this is indeed a path homotopy. □

From now on we can be lazy and omit parentheses when talking about path homotopy classes (but not when talking about loops themselves).

LEMMA (Identity). *Let f be a path in X which begins at x_0 and ends at x_1. Then* $[f][e_{x_1}] = [f] = [e_{x_0}][f]$.

PROOF. We prove that $f * e_{x_1} \sim f$. The other case is similar.



In order to define a homotopy, at t, we will "do" f for $s \in [0, t(1) + (1-t)\frac{1}{2}]$ or, by simplification, $s \in [0, \frac{t+1}{2}]$ and $e_{x_1}(t)$ for $s \in [\frac{t+1}{2}, 1]$.

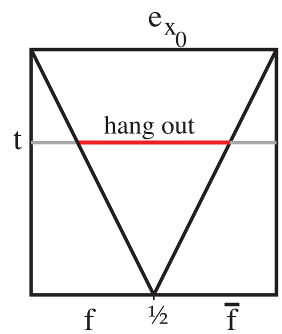
Define $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} f\left(\frac{2s}{t+1}\right) & s \in \left[0, \frac{t+1}{2}\right] \\ e_{x_1} & s \in \left[\frac{t+1}{2}, 1\right] \end{cases}$$

We leave it as an exercise to check that this is indeed a path homotopy.□

We've now shown that $\pi_1(X, x_0)$ is closed, has an identity, and the operation is associative, so just showing that inverses exist proves that it's a group.

LEMMA (Inverses). *Let f be a path in X from x_0 to x_1. Then* $[f][\bar{f}] = [e_{x_0}]$ *and* $[\bar{f}][f] = [e_{x_1}]$.



The idea of this proof is different from the usual algorithm. We don't want to increase the speed of $f * \bar{f}$ and wait at x_0, since this will never give us e_{x_0}. Rather, we keep f at the same speed but do less of it, then hang out whenever we are, then return via $\bar{f}$.

Proof. We show only that $f * \bar{f} \sim e_{x_0}$, and the other case follows similarly. So at t we do f at the usual speed for $s \in [0, t(0) + (1-t)\frac{1}{2}] = [0, \frac{1-t}{2}]$. Then hang out where we are for $s \in [\frac{1-t}{2}, t + (1-t)\frac{1}{2}] = [\frac{1-t}{2}, \frac{1+t}{2}]$. Then do $\bar{f}$ for the remaining time. Note we don't shift f because we want it to always start at 0, and we don't shift $\bar{f}$ because we want it to always end at 1. Also both go at the usual speed of 2.

Define $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} f(2s) & s \in \left[0, \frac{1-t}{2}\right] \\ f(1-t) & s \in \left[\frac{1-t}{2}, \frac{1+t}{2}\right] \\ \bar{f}(2s-1) & s \in \left[\frac{1+t}{2}, 1\right] \end{cases}$$

Again we leave it as an exercise to check this is a path homotopy. □

It follows that $\pi_1(X, x_0)$ is a group. Observe that the set of paths does not have a group structure, since there is no definition of multiplication between arbitrary paths.

Example. Let $X = \mathbb{R}^n$ and x_0 be a point in X. Then $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$ since the straight line homotopy sends every loop to e_{x_0}.

Definition. Let (X, F_x) be path connected and suppose for all $x_0 \in X$,

$$\pi_1(x, x_0) = \langle [e_{x_0}] \rangle.$$

Then we say that X is **simply connected**.

Example. $\mathbb{R}^n$ is simply connected.

Example. What about $\mathbb{Q}$? $\mathbb{Q}$ is *not* path connected so it can't be simply connected. But $\pi_1(Q, x_0) = \langle [e_{x_0}] \rangle$, as e_{x_0} is the only path from x_0: all others must pass through irrationals, and hence cannot be contained in $\mathbb{Q}$.

The following theorem relates the fundamental group at a point within a path component to the fundamental group of that point in the ambient space:

Theorem. *Let A be a path component of a topological space X, and let $x_0 \in A$. Then:*

$$\pi_1(A, x_0) \cong \pi_1(X, x_0)$$

(Note that $\cong$ denotes a group isomorphism and not a homeomorphism)

Question: What's the definition of a group isomorphism?

Before proving the theorem, we will cover a quick non-example.

Consider the circle S^1as a subspace of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is a simply connected space, the fundamental group at every point is trivial. On the other hand, picking some point $x_0 \in S^1$, the loop around the circle cannot be deformed in S^1 to the point x_0 (though we have not yet proved this). So $\pi_1(S^1, x_0)$ is non-trivial and hence not isomorphic to $\pi_1(\mathbb{R}^2, x_0)$.

At first this may seem like it gives a counterexample to the theorem. But no matter how we embed S^1 in $\mathbb{R}^2$, we see that S^1 is not a distinct path component. So the theorem does not apply. Intuitively, by embedding S^1 in $\mathbb{R}^2$, the interior of the circle is part of $\mathbb{R}^2$, so we can deform a loop around S^1 based at x_0 to the trivial loop by "pulling" the loop through the middle of the circle, which we could not do when S^1 was considered as a space in its own right.

We now prove the theorem:

Proof. Recall that A is a path component of X. Define $\varphi : \pi_1(A, x_0) \to \pi_1(X, x_0)$ by:

$$\varphi([f]_A) = [f]_X$$

We claim that φ is a *group isomorphism*, i.e. that φ is a bijection and a group homomorphism. In other words, we need to show that φ is injective, surjective and that if $a, b \in \pi_1(A, x_0)$, then $\varphi(ab) = \varphi(a)\varphi(b)$.

Well Defined: Before we actually prove that φ satisfies the properties of an isomorphism, we have to show φ is well defined because φ is defined in terms of equivalence classes. First, let f, g be loops in A based at $x_0 \in A$ such that $f \sim_A g$. Thus there exists a path homotopy, $F : I \times I \to A$, from f to g. Recall that the inclusion map $i : A \to X$ is defined as the identity map on X restricted to A. Then i is trivially continuous. Thus we can extend F to the continuous map $i \circ F : I \times I \to X$, and it is easy to see that $i \circ F$ is a path homotopy in X from f to g, so $f \sim_X g$. Thus if $[f]_A = [g]_A$, then $\varphi([f]_A) = \varphi([g]_A)$, and hence φ is well defined.

Injective: Suppose $[f]_A, [g]_A \in \pi_1(A, x_0)$ such that $\varphi([f]_A) = \varphi([g]_A)$. Then $[f]_X = [g]_X$. Hence there exists $F : I \times I \to X$, a path homotopy from f to g in X. We show as follows that the image of F is contained in A. Observe that $I \times I$ is path connected and F is continuous. Thus $F(I \times I)$ is path connected, and hence $F(I \times I)$ is contained in a single path component of X. Now since $F(0,0) = x_0 \in A$, $F(I \times I) \subseteq A$. Therefore, F is a homotopy from f to g in A. Consequently, $f \sim_A g$, and hence $[f]_A = [g]_A$. Thus φ is injective.

Surjective: Let $[f]_X \in \pi_1(X, x_0)$, so f is a loop in X based at x_0. Hence, $f : I \to X$ is a path containing x_0. Since A is a path component, $f(I) \subseteq A$. Consequently, f is a loop in A based at x_0. Thus $\varphi([f]_A) = [f]_X$ and φ is surjective.

Homomorphism: Let $[f]_A, [g]_A \in \pi_1(A, x_0)$. We see that:

$$\varphi([f]_A[g]_A) = \varphi([f * g]_A) = [f * g]_X = [f]_X[g]_X = \varphi([f]_A)\varphi([g]_A)$$

and φ satisfies the definition of a homomorphism.

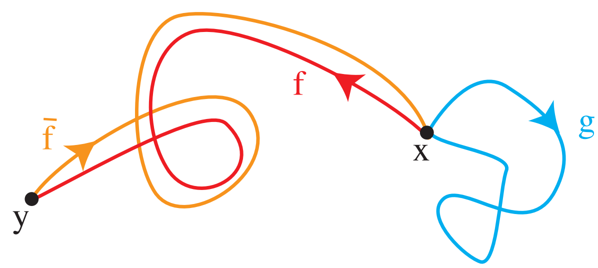
We have proved that φ is a bijective, homomorphism, and is therefore an isomorphism between $\pi_1(A, x_0)$ and $\pi_1(X, x_0)$. $\square$

We now wish to prove that within a path component the base point doesn't matter:

Theorem. *Let X be a topological space and let $x, y \in X$. Suppose $f : I \to X$ is a path from x to y, then $\pi_1(X, x) \cong \pi_1(X, y)$.*

First we define a map which we will use again later in the course:

Definition. Define $u_f : \pi_1(X, x) \to \pi_1(X, y)$ by $u_f([g]) = [\bar{f} * g * f]$.[3]



We now prove the theorem by showing that u_f is an isomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$:

Proof. Again, we need to show that u_f is a bijective homomorphism. As usual for functions defined in terms of equivalence classes, we need to show that u_f is actually well defined.

Well defined: Suppose that g, h are loops in X based at x such that $g \sim h$, so there exists $F : I \times I \to X$, a path homotopy from g to h in X. We want to show that:

$$\bar{f} * g * f \sim \bar{f} * h * f$$

[3]Technically, $\bar{f} * g * f$ should have parentheses, but we dispense with these because we previously proved that the invisible operation is associative.

Note,we really should have parentheses. But we omit them because we know that within the square brackets they aren't necessary. Since we know trivially $f \sim f$, $g \sim h$ and $\bar{f} \sim \bar{f}$, using our Important Lemma for products of path homotopy classes, it is easy to see that $\bar{f} * g * f \sim \bar{f} * h * f$, so u_f is well defined.

Injective: Suppose g, h are loops in X based at x such that $u_f([g]_X) = u_f([h]_X)$. Therefore:

$$[\bar{f} * g * f] = [\bar{f} * h * f] \Rightarrow \bar{f} * g * f \sim \bar{f} * h * f$$

so by our Important Lemma for products and our inverse/identity lemmas, we have that $g \sim h$ and u_f is injective.

Surjective: Let $[g] \in \pi_1(X, y)$. Then $[f * g * \bar{f}] \in \pi_1(X, x)$, and $u_f([f * g * \bar{f}]) = [\bar{f} * (f * g * \bar{f}) * f]$, which by associativity, inverses, and identity, is precisely $[g]$. Therefore u_f is onto.

Homomorphism: Let $g, h \in \pi_1(X, x)$. Then

$$\begin{aligned} u_f([g])u_f([h]) &= [\bar{f} * g * f][\bar{f} * h * f] \\ &= [\bar{f} * g * h * f] \\ &= u_f([g * h]) \end{aligned}$$

Therefore, u_f is a group isomorphism. $\square$

16.2. Induced Maps. Our overall goal right now is to show that the fundamental group is a particular type of topological property known as a *topological invariant.* In general, a *topological invariant* is a function on topological spaces such that when applying the function to homeomorphic spaces you get the same value. So we need to prove that homeomorphic spaces have isomorphic fundamental groups. To do this we introduce induced maps.

Definition. Let $\varphi : X \to Y$ be continuous, and $\varphi(x_0) = y_0$. Define the map

$\varphi_* : \pi_1(X, x_0) \to \pi_1(X, y_0)$ by $\varphi_*([f]_X) = [\varphi(f)]_Y$. We say that φ_* is *induced* by φ.

Small Fact (about induced maps). Let $\varphi : X \to Y$ and $\varphi(x_0) = y_0$. Then φ_* is well defined.

Proof. Let f and g be loops in X based at x_0 such that $f \sim g$. Then there exists $F : I \times I \to X$, a path homotopy from f to g. Consider $\varphi(F) : I \times I \to Y$. We see that φ is a composition of continuous functions, so is itself continuous.

Also, observe that

$$\begin{aligned}\varphi(F(s,0)) &= \varphi \circ f(s)\\ \varphi(F(s,1)) &= \varphi \circ g(s)\\ \varphi(F(0,t)) &= \varphi(x_0) = y_0\\ \varphi \circ F(1,t) &= \varphi(x_0) = y_0\end{aligned}$$

Thus $\varphi(F)$ is a path homotopy between $\varphi \circ g$ and $\varphi \circ f$. So $\varphi \circ g \sim \varphi \circ f$. Hence φ_* is well defined. $\square$

LEMMA. *Let $\varphi : X \to Y$ be continuous and $\varphi(x_0) = y_0$. Then φ_* is a homomorphism.*

PROOF. Let $[f], [g] \in \pi_1(X, x_0)$. We want to show that $\varphi_*([f]_X[g]_X) = \varphi_*([f]_X)\varphi_*([g]_X)$.

Observe that

$$\begin{aligned}\varphi_x([f]_X[g]_X) &= \varphi_*([f*g]_X) = [\varphi(f*g)]_Y\\ \varphi \circ (f*g)(x) &= \varphi_* \begin{cases} f(2s), & s \in [0,1/2]\\ g(2s-1) & s \in [1/2,1]\end{cases}\\ &= \begin{cases} \varphi \circ f(2s) & s \in [0,1/2]\\ \varphi \circ g(2s-1) & s \in [1/2,1]\end{cases}\\ &= (\varphi \circ f) * (\varphi \circ g)(s).\end{aligned}$$

So $\varphi \circ (f*g) = (\varphi \circ f) * (\varphi \circ g)$ and, in particular,

$$\begin{aligned}[\varphi(f*g)]_Y &= [(\varphi \circ f) * (\varphi \circ g)]_Y\\ &= [\varphi \circ f]_Y[\varphi \circ g]_Y\\ &= \varphi_*([f]_X)\varphi_*([g]_X)\end{aligned}$$

This concludes the proof of the lemma. $\square$

THEOREM. *Let $\varphi : X \to Y$ be a homeomorphism and $\varphi(x_0) = y_0$. Then φ_* is an isomorphism.*

With the previous lemma we showed that φ_* is a homomorphism. It remains to show that φ_* is a bijection.

PROOF. **Injective:** Let $[f]_X, [g]_X \in \pi_1(X, x_0)$ such that $\varphi_*([f]_X) = \varphi_*([g]_X)$. Then by definition of φ_*, we know that $[\varphi \circ f]_Y = [\varphi \circ g]_Y$.

Since $\varphi \circ f \sim_Y \varphi \circ g$, there exists a path homotopy F from $\varphi \circ f$ to $\varphi \circ g$. Also note that, as φ is a homeomorphism, $\varphi^{-1} : Y \to X$ is continuous.

Hence $\varphi^{-1} \circ F : I \times I \to Y$ is a path homotopy from $\varphi^{-1} \circ \varphi \circ f$ to $\varphi^{-1} \circ \varphi \circ g$. This is a path homotopy from f to g.

Surjective: Recall that $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Let $[f] \in \pi_1(Y, y_0)$. Then $[\varphi^{-1}(f)]_X \in \pi_1(X, x_0)$, because φ^{-1} is continuous.

$\varphi_*([\varphi^{-1}(f)]_X) = [\varphi \circ \varphi^{-1}(f)]_Y$, and as φ is a bijection, this is $[f]_Y$.

This illustrates that φ_* is bijective, and hence is an isomorphism between X and Y. $\square$

Small Fact (about induced homomorphisms). The following are true:

1) If $\varphi : X \to Y$ and $\psi : Y \to Z$ are continuous, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$

2) If $i : X \to X$ is the identity, then i_* is the identity isomorphism

3) Let $\varphi : X \to Y$ be continuous and f a path in X from p to q. Then $\varphi_* \circ u_f = u_{\varphi(f)} \circ \varphi_*$. (recall, $u_f : \pi_1(X, p) \to \pi_1(X, q)$ is defined by $u_f([g]) = [\bar{f} * g * f]$)

Proof. 1) Note that $(\psi \circ \varphi)_* : \pi_1(X, x_0) \to \pi_1(Z, (\psi \circ \varphi)(x_0))$, is a mapping from equivalence classes of loops in X based at x_0 to equivalence classes of loops in Z based at $(\psi \circ \varphi)(x_0)$. Let $[f]_X \in \pi_1(X, x_0)$. We then have that $(\psi \circ \varphi)_*([f]_X) = [(\psi \circ \varphi)(f)]_Z$. Similarly, we note that $(\psi_* \circ \varphi_*)([f]_X) = \psi_*([\varphi \circ f]_Y) = [\psi \circ \varphi \circ f]_Z = [(\psi \circ \varphi)(f)]_Z$.

2) Note that the induced homomorphism is $i_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$. Let $[f]_X \in \pi_1(X, x_0)$. Then $i_*([f]_X) = [i(f)]_X = [f]_X$.

3) We want to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,p) & \xrightarrow{u_f} & \pi_1(X,q) \\ \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* \\ \pi_1(Y,\varphi(p)) & \xrightarrow{u_{\varphi(f)}} & \pi_1(Y,\varphi(q)) \end{array}$$

Let $[g]_X \in \pi_1(X, p)$. Then note that

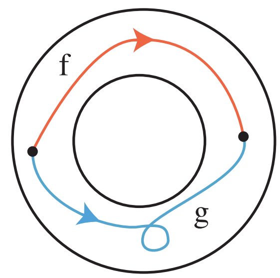
$$\begin{aligned} (\varphi_* \circ u_f)([g]_X) &= \varphi_*([\bar{f} * g * f]_X) \\ &= [\varphi \circ (\bar{f} * g * f)]_Y \\ &= [(\varphi \circ \bar{f}) * (\varphi \circ g) * (\varphi \circ f)]_Y \end{aligned}$$

To see that $\varphi \circ \bar{f} = \overline{\varphi \circ f}$ note that $(\varphi \circ \bar{f})(s) = \varphi \circ \bar{f}(s) = \varphi(f(1-s)) = (\varphi \circ f)(1-s) = \overline{\varphi \circ f}(s)$. So continuing the above expression, we have that:

$$\begin{aligned} [(\varphi \circ \bar{f}) * (\varphi \circ g) * (\varphi \circ f)]_Y &= [(\overline{\varphi \circ f}) * (\varphi \circ g) * (\varphi \circ f)]_Y \\ &= u_{\varphi \circ f}([\varphi \circ g]_Y) \\ &= (u_{\varphi(f)} \circ \varphi_*)([g]_X) \end{aligned}$$

We then conclude that $\varphi_* \circ u_f = u_{\varphi(f)} \circ \varphi_*$. $\square$

Lemma. *Suppose X is path connected, and $x_0 \in X$. Then $\pi_1(X, x_0)$ is trivial if and only if $\forall p, q \in X$ and paths f, g in X from p to q, then $f \sim g$.*



Proof.
($\Rightarrow$) Suppose that $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$. Let $p, q \in X$, and f, g paths in X from p to q. Then $f * \overline{g}$ is a loop based at p. So $\pi_1(X, p) \cong \pi_1(X, x_0)$ by an earlier theorem, from which we can see that $f * \overline{g} \sim e_p$. Using our multiplication and inverse lemmas for path multiplication, we conclude that $f \sim g$.

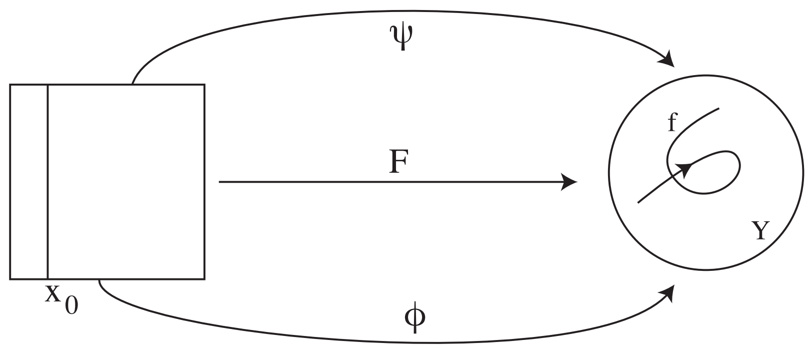
($\Leftarrow$) Suppose that $\forall p, q \in X$ and paths f, g from p to q, we have that $f \sim g$. Let $p = q = x_0$, let $f = e_{x_0}$, and let g be a loop in X based at x_0. Then $g \sim e_{x_0}$. Hence, $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$.
$\square$

We will skip the following section if we are tight for time

16.3. Homotopy Equivalence and Fundamental group. We'd like to prove that path connected spaces spaces that are homotopy equivalent have isomorphic fundamental groups. First, we'll need a technical lemma.

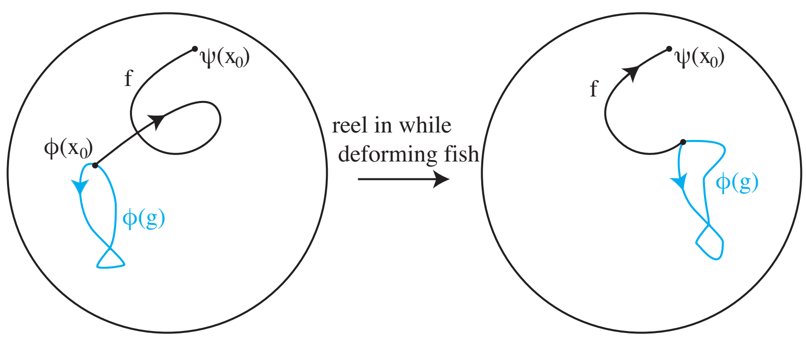
Lemma (Fishing Lemma). *Let $\varphi, \psi : X \to Y$ be continuous, and $\varphi \simeq \psi$ by a homotopy F. Let $x_0 \in X$, and a path $f : I \to Y$ be given by $f(t) = F(x_0, t)$. Then $u_f \circ \varphi_* = \psi_*$.*

Note f is the path taken by the image of the vertical segment at x_0. Also, $u_f : \pi_1(Y, \varphi(x_0)) \to \pi_1(Y, \psi(x_0))$ by $u_f([h]_Y) = [\bar{f} * h * f]_Y$.

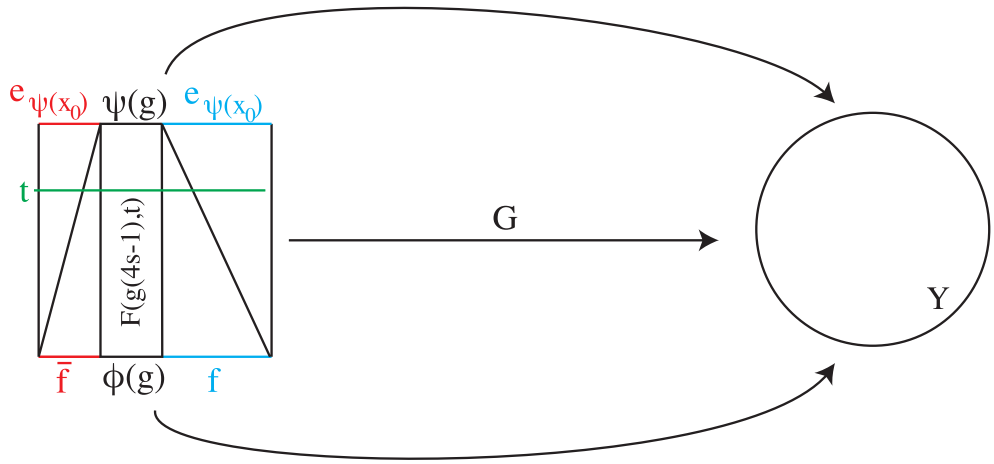


PROOF. I will talk about this proof before we do it in the round.

Let $[g] \in \pi_1(X, x_0)$. Then $u_f \circ \varphi_* ([g]_X) = u_f([\varphi \circ g]_X) = [\bar{f} * (\varphi \circ g) * f]_Y$. We want to show that this equals $\psi_*([g]_X) = [\psi \circ g]_Y$. In particular, we want to show $\bar{f} * \varphi(g) * f \sim_Y \psi(g)$. We know there is a homotopy (but not a path homotopy) from $\varphi(g)$ to $\psi(g)$, which takes the point $\varphi(x_0)$ along the path f to bring it to $\psi(x_0)$. We can think of f as a "fishing rod", and to get a path homotopy from $\bar{f} * \varphi(g) * f$ to $\psi(g)$, we reel in the "fish" $\varphi(g)$ along the fishing rod while deforming the fish according to the homotopy from $\varphi(g)$ to $\psi(g)$.



However, rather than finding a path homotopy to show $\bar{f} * \varphi(g) * f \sim_Y \psi(g)$, we define a path homotopy to show $\big(\bar{f} * \varphi(g)\big) * f \sim_Y \big(e_{\psi(x_0)} * \psi(g)\big) * e_{\psi(x_0)}$. We do this so that the time segments on the top and the bottom are the same.



From the picture we see that at time t we hang out at $\psi(x_0)$ for $s \in [0, \frac{t}{4}]$, then do part of $\bar{f}$ for $s \in [\frac{t}{4}, \frac{1}{4}]$ at speed 4 shifted so that it starts at $s = \frac{t}{4}$ and ends at $s = \frac{1}{4}$. Then for $s \in [\frac{1}{4}, \frac{1}{2}]$, we do $F(g(s), t)$ at s speed 4 shifted so that it starts at $s = \frac{1}{4}$ and ends at $s = \frac{1}{2}$. Then do part of f at speed 2 for $s \in [\frac{1}{2}, \frac{2-t}{2}]$ shifted so that it starts at $s = \frac{1}{2}$ and ends at $\frac{2-t}{2}$. Finally we hang out at $\psi(x_0)$ for $s \in [\frac{2-t}{2}, 1]$.

We define $G : I \times I \to Y$ by

$$G(s,t) = \begin{cases} e_{\psi(x_0)} & s \in \left[0, \frac{t}{4}\right] \\ \bar{f}(4s - t) & s \in \left[\frac{t}{4}, \frac{1}{4}\right] \\ F(g(4s-1), t) & s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ f(2s - 1 + t) & s \in \left[\frac{1}{2}, \frac{2-t}{2}\right] \\ e_{\psi(x_0)} & s \in \left[\frac{2-t}{2}, 1\right] \end{cases}$$

We will NOT check in class that this works, because it's tedious. The proof for you to read is in blue.

Continuous. G is defined differently over the five closed regions. Over each region, G is the composition of continuous functions and hence continuous. For G to be continuous everywhere, the value of G at the intersection of any two adjoining regions must agree.

- $s = \frac{t}{4}$: $\bar{f}(4 \cdot \frac{t}{4} - t) = \bar{f}(0) = \psi(x_0) = e_{\psi(x_0)}$.
- $s = \frac{1}{4}$: $\bar{f}(4 \cdot \frac{1}{4} - 1) = \bar{f}(1 - t) = f(t)$.
  $F(g(4 \cdot \frac{1}{4} - 1), t) = F(g(0), t) = F(x_0, t) = f(t)$.
- $s = \frac{1}{2}$: $F(g(4 \cdot \frac{1}{2} - 1), t) = F(g(1), t) = F(x_0, t) = f(t)$.
  $f(2\frac{1}{2} - 1 + t) = f(t)$.

- $s = \frac{2-t}{2}$: $f(2 \cdot \left(\frac{2-t}{2}\right) - 1 + t) = f(1) = \psi(x_0) = e_{\psi(x_0)}$.

Hence by the Pasting Lemma, the function G is continuous.

Homotopy. Consider $G(s, 0)$:

$$G(s,0) = \begin{cases} e_{\psi(x_0)} & s \in [0,0] \\ \bar{f}(4s) & s \in \left[0, \frac{1}{4}\right] \\ F(g(4s-1), 0) & s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ f(2s-1) & s \in \left[\frac{1}{2}, 1\right] \\ e_{\psi(x_0)} & s \in [1,1] \end{cases}$$

Thus, $G(s, 0) = \bar{f} * \varphi(g) * f$. Consider $G(s, 1)$:

$$G(s,1) = \begin{cases} e_{\psi(x_0)} & s \in \left[0, \frac{1}{4}\right] \\ \bar{f}(4s-1) & s \in \left[\frac{1}{4}, \frac{1}{4}\right] \\ F(g(4s-1), 1) & s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ f(2s) & s \in \left[\frac{1}{2}, \frac{1}{2}\right] \\ e_{\psi(x_0)} & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Thus, $G(s, 1) = \left(e_{\psi(x_0)} * \psi(g)\right) * e_{\psi(x_0)}$.

Path. $G(0, t) = \psi(x_0)$, and $G(1, t) = \psi(x_0)$.

Hence G is a path homotopy from $\left(\bar{f} * \varphi(g)\right) * f$ to $\left(e_{\psi(x_0)} * \psi(g)\right) * e_{\psi(x_0)}$. Therefore, $u_f(\varphi([g]) = \psi([g])$ for all $[g] \in \pi_1(X, x_0)$ and $u_f \circ \varphi_* = \psi_*$. $\square$

We now use the Fishing Lemma to prove that homotopy equivalent spaces have isomorphic fundamental groups.

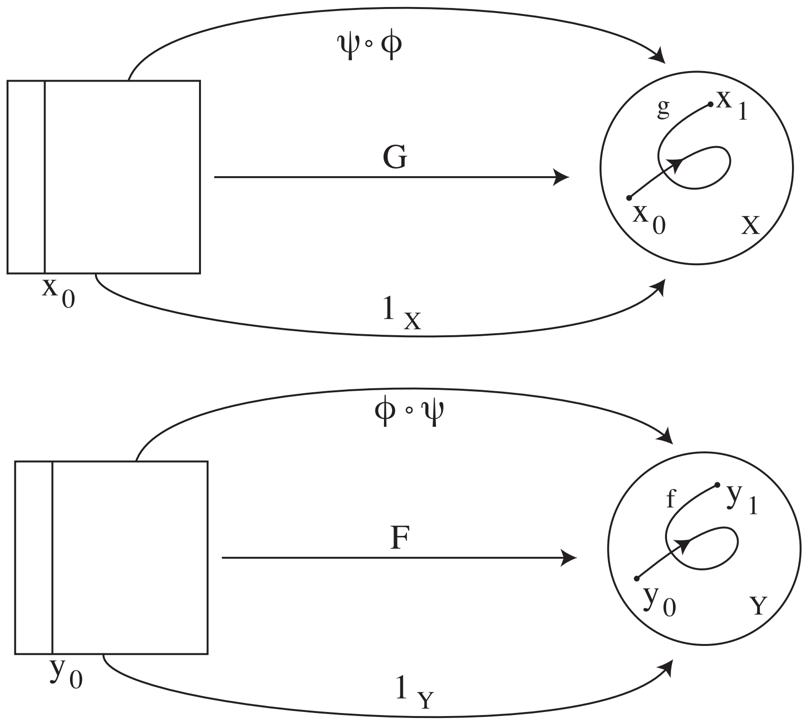
Corollary. *Let $\varphi : X \to Y$ and $\psi : Y \to X$ be continuous such that $\varphi \circ \psi \simeq 1_Y$ and $\psi \circ \varphi \simeq 1_X$. Let $\varphi(x_0) = y_0$. Then $\varphi_{x_0 *} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Note that given $\varphi : X \to Y$, there can be many different induced homomorphisms φ_* depending on our choice of base point for the fundamental group of X. For this reason, we use the notation $\varphi_{x_0 *}$ to make it clear that we have chosen x_0 as the base point.

The proof of this Corollary is a bit confusing, so I'll do it myself.

Proof. We have already proven that $\varphi_{x_0 *}$ is a homomorphism. For $\varphi_{x_0 *}$ to be an isomorphism, we only need to prove that it's a bijection. We will need to use the Fishing Lemma, which we set up as follows.

Let G be a homotopy in X from 1_X to $\psi \circ \varphi$, and let F be a homotopy in Y from 1_Y to $\varphi \circ \psi$. Now define a path g in X by $g(t) = G(x_0, t)$, and a path f in Y by $f(t) = F(y_0, t)$. Then g is a path from x_0 to $x_1 = \psi \circ \varphi(x_0) = \psi(y_0)$ and f is a path from y_0 to $y_1 = \varphi \circ \psi(y_0) = \varphi(x_1)$.



Now by the Fishing Lemma, $u_f \circ 1_{Y_*} = (\varphi \circ \psi)_*$ and $u_g \circ 1_{X_*} = (\psi \circ \varphi)_*$. However, we need to specify the base point in the domain of each of these induced homomorphisms. Since f is a path from y_0 to y_1, we have $u_f \circ 1_{Y_*} = (\varphi \circ \psi)_{y_0*}$; and since g is a path from x_0 to x_1, we have $u_g \circ 1_{X_*} = (\psi \circ \varphi)_{x_0*}$.

We can now apply our small facts to get $u_f \circ 1_{Y_*} = u_f$ and $(\varphi \circ \psi)_{y_0*} = \varphi_{x_1*} \circ \psi_{y_0*}$. Note the subscript on φ is x_1 because $\psi(y_0) = x_1$. Hence $u_f = \varphi_{x_1*} \circ \psi_{y_0*}$. Similarly, $u_g = \psi_{y_0*} \circ \varphi_{x_0*}$. Note we previously showed that u_f and u_g are isomorphisms.

We will show that ψ_{y_0*} is an isomorphism and then use this to show that φ_{x_0*} is an isomorphism. Since $u_g = \psi_{y_0*} \circ \varphi_{x_0*}$ is an isomorphism it is onto. It follows that ψ_{y_0*} is onto, since the image of $u_g = \psi_{y_0*} \circ \varphi_{x_0*}$ is contained in the image of ψ_{y_0*}. Also since $u_f = \varphi_{x_1*} \circ \psi_{y_0*}$ is an isomorphism it is one to one. It follows that ψ_{y_0*} must be one to one, since if $\psi_{y_0*}([h_1]) = \psi_{y_0*}([h_2])$ then $u_f([h_1]) = u_f([h_2])$. Thus ψ_{y_0*} is an isomorphism.

Now $u_f = \varphi_{x_1*} \circ \psi_{y_0*}$ implies that $u_f \circ \psi_{y_0*}^{-1} = \varphi_{x_1*}$ is an isomorphism. $\square$

It follows from this corollary that the cylinder $S^1 \times I$ and the circle S^1 have isomorphic fundamental groups. But we still can't prove that this fundamental group is $\mathbb{Z}$.

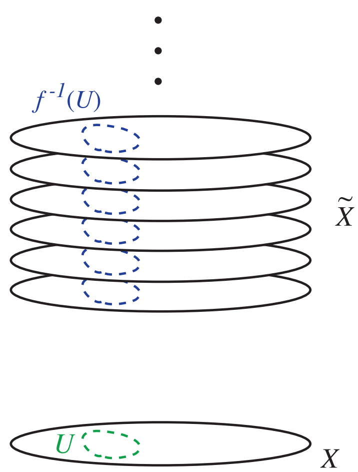
17. Covering Maps

Now the flavor of the course returns to what is was before we started homotopy.

Our ultimate goal is to find a space with non-trivial fundamental group. But first we need to understand covering spaces.

DEFINITION. Let X and $\widetilde{X}$ be topological spaces, and $p : \widetilde{X} \twoheadrightarrow X$ be a continuous surjection. An open set $U \subseteq X$ is said to be **evenly covered** by p if $p^{-1}(U)$ is the disjoint union of open sets V_α, $\alpha \in A$ for some index set A, such that for all $\alpha \in A$, $p \mid V_\alpha \to U$ is a homeomorphism. In this case, we say that each V_α is a **sheet** covering U.

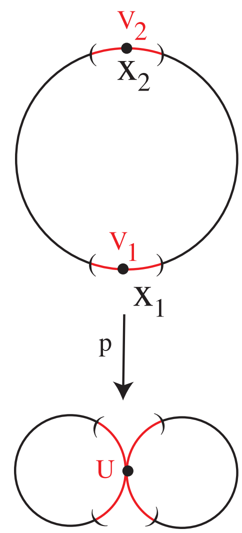
EXAMPLE. Let $X = D^2$ with the usual topology, and $\widetilde{X} = D^2 \times \mathbb{N}$ with the usual product topology. Define $p : \widetilde{X} \to X$ by $p(x, n) = x$. We can take U to be any open set in X; $p^{-1}(U) = \displaystyle\bigcup_{i=1}^{\infty} p^{-1}(U) \cap V_i$, where $V_i = D^2 \times i$.



But this is a boring example.

EXAMPLE (a non-example). Let $\widetilde{X} = S^1$, and let $X = S^1 \vee S^1$, be the *wedge* of two circles. That is, X is two copies of S^1, which agree at a point.

Let x_1 and x_2 be a pair of antipodal points in S^1. Let $\sim$ be the equivalence relation on S^1 given by $x \sim y$ if and only if $x, y \in \{x_1, x_2\}$ or $x = y$. Let p the quotient map from $\widetilde{X}$ to X, corresponding to this relation, and let U be an open set in X containing $p(x_1)$ as shown.



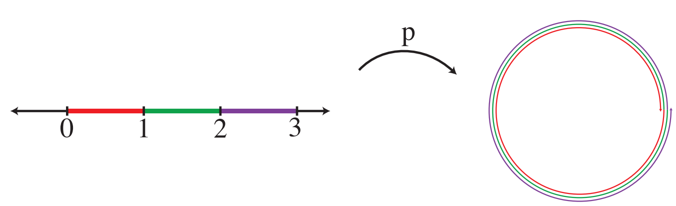
Is U evenly covered? The answer is no. To see why, note that $p^{-1}(U) = V_1 \cup V_2$, where V_1, V_2 are disjoint open sets in $\widetilde{X}$. But $p|V_1 : V_1 \to U$ is not a homeomorphism because it is not onto.

Definition. Let $p : \widetilde{X} \twoheadrightarrow X$ be a continuous surjection. Suppose for all $x \in X$, there exists an evenly covered open set U containing x. We say p is a **covering map**, with **covering space** $\widetilde{X}$, and **base space** X.

Note the example of the wedge of two circles shows that not all quotient maps are covering maps.

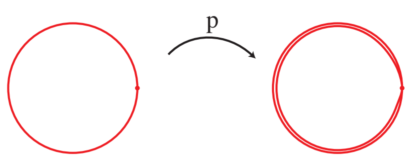
Example (another non-example). Let $\widetilde{X} = \mathbb{R}^2$, $X = \mathbb{R}$, and $p : \mathbb{R}^2 \to \mathbb{R}$ be defined by $p(x, y) = x$. Then for each open $U \subseteq X$, $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$. We can think of $p^{-1}(U)$ as a vertical stack of uncountably many copies of U. For each $\alpha \in A$, $p \mid V_\alpha : V_\alpha \to U$ is a homeomorphism. However, each V_α is not open in $\widetilde{X}$. Thus U is not evenly covered.

Example (Important). Let $p : \mathbb{R} \to S^1$ be defined by $p(x) = (\cos(2\pi x), \sin(2\pi x))$. (The "slinky" space.)



For each $s \in S^1$, an "open interval" around s is evenly covered, and so $\mathbb{R}$ is a covering space.

EXAMPLE. $\widetilde{X} = S^1$, $X = S^1$ by $p : \widetilde{X} \to X$ is $p((\cos(2\pi x), \sin(2\pi x)) = (\cos(4\pi x), \sin(4\pi x))$. This is also a covering map.

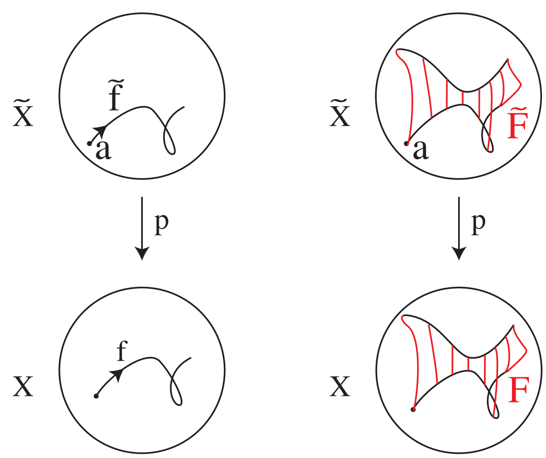


I list here two major theorems which we will prove later (if there is time). You will need these theorems to do the homework.

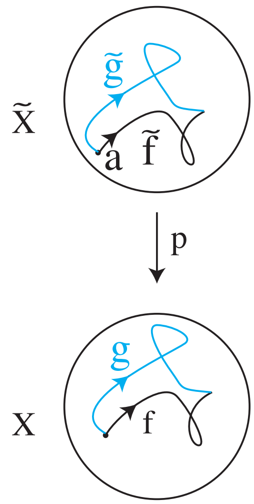
THEOREM (Homotopy Path Lifting Theorem). *Let $p : \widetilde{X} \to X$ be a covering map. Then,*

(1) *Given a path f in X and $a \in \widetilde{X}$ such that $p(a) = f(0)$, then $\exists!$ (exists unique) path $\widetilde{f}$ in $\widetilde{X}$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(0) = a$.*

(2) *Given a continuous map $F : I \times I \to X$ and $a \in \widetilde{X}$ with $p(a) = F(0,0)$, $\exists!$ continuous map $\widetilde{F} \colon I \times I \to \widetilde{X}$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(0,0) = a$.*



THEOREM (Monodromy Theorem). *Let $p : \widetilde{X} \to X$ be a covering map, and let $a \in \widetilde{X}$. Let $x_1, x_2 \in X$. Suppose that $p(a) = x_1$, and that f, g are paths in X from x_1 to x_2 such that $f \sim g$. Let $\widetilde{f}, \widetilde{g}$ be the unique paths beginning at a such that $p \circ \widetilde{f} = f$ and $p \circ \widetilde{g} = g$. Then $\widetilde{f}(1) = \widetilde{g}(1)$ and $\widetilde{f} \sim \widetilde{g}$.*



Now we return to proving things as we go along.

Lemma (Important Lemma on Covering Maps). *Let $p : \widetilde{X} \to X$ be a covering map. Let $x \in X$. Then the subspace topology on $p^{-1}(\{x\})$ is the discrete topology.*

This lemma tells us that examples like the projection map $p : \mathbb{R}^2 \to \mathbb{R}$ are not covering maps.

Proof. Let $y \in p^{-1}(\{x\})$. We want to show that $\{y\}$ is open in $p^{-1}(\{x\})$. Since p is a covering map, there exists an evenly covered open set U containing x. Then $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$, where the V_α are pairwise-disjoint open sets such that $p \mid V_\alpha : V_\alpha \to U$ is a homeomorphism for every $\alpha \in A$. Hence, there exists $\alpha_0 \in A$ such that $y \in V_{\alpha_0}$. Now, V_{α_0} is open in $\widetilde{X}$, and $y \in V_{\alpha_0} \cap p^{-1}(\{x\})$.

We want to show that $V_{\alpha_0} \cap p^{-1}(\{x\}) = \{y\}$. Let $y' \in V_{\alpha_0} \cap p^{-1}(\{x\})$ be given. We know that $p \mid V_{\alpha_0}$ is a homeomorphism, so it is injective. Since $p(y') = x$, and $p(y) = x$, we see that $y = y'$. Thus, $\{y\} = V_{\alpha_0} \cap p^{-1}(\{x\})$. So $\{y\}$ is open in $p^{-1}(\{x\})$ with the subspace topology. This completes the proof. $\square$

The take-home message is that in a covering space, points in the pre-image of a single point are "spread out."

Not all quotient maps are covering maps as we have seen above. But we will prove that all covering maps are quotient maps.

Theorem. *Let $p : \widetilde{X} \to X$ be a covering map. Then*

(1) *p is an open map (this is not true for all quotient maps)*

(2) *X is a quotient space, and p is a quotient map.*

PROOF. 1) Let U be open in $\widetilde{X}$. We want to show that $p(U)$ is open. Let $x \in p(U)$. We want to show that there is an open set W such that $x \in W \subseteq p(U)$.

There exists an evenly covered open set V containing x. Then $p^{-1}(V) = \bigcup_{\alpha \in A} V_\alpha$ such that the V_α are disjoint open sets and $p \mid V_\alpha$ is a homeomorphism for all $\alpha \in A$. Since $x \in p(U)$, there exists $y \in U$ such that $p(y) = x$, and hence there exists $\alpha_0 \in A$ such that $y \in V_{\alpha_0}$.

Because U and V_{α_0} are both open, $U \cap V_{\alpha_0}$ is open in $\widetilde{X}$. Since $p \mid V_{\alpha_0} : V_{\alpha_0} \to V$ is a homeomorphism,

$$\begin{aligned} p(U \cap V_{\alpha_0}) &= p(V_{\alpha_0}) \cap p(U) \\ &= V \cap p(U) \end{aligned}$$

Furthermore, since $p \mid V_{\alpha_0} : V_{\alpha_0} \to V$ is a homeomorphism, $p(U \cap V_{\alpha_0}) = V \cap p(U)$ is open in V. Now since V is open in X, $V \cap p(U)$ is open in X. Because our x is in both V and $p(U)$, $x \in V \cap p(U) \subseteq p(U)$. That is, x is an element of an open set contained in $p(U)$. Hence, $p(U)$ is open and p is an open map.

2) To show X has the quotient topology with respect to p, we want to show that $F_X = \{U \subseteq X \mid p^{-1}(U) \in F_X\}$.

($\subseteq$) Let $V \in F_X$. Because p is continuous, $p^{-1}(V) \in F_{\widetilde{X}}$. Thus, $V \in \{U \subseteq X \mid p^{-1}(U) \in F_X\}$.

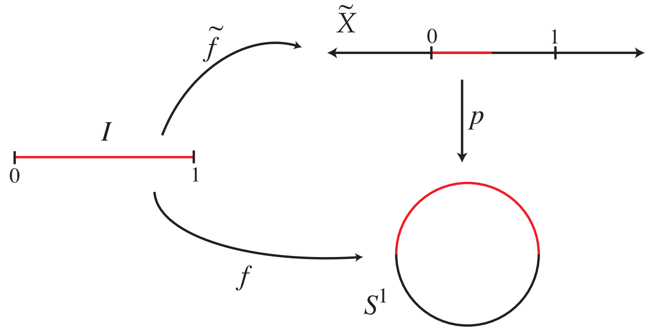
($\supseteq$) Let $U \subseteq X$ such that $p^{-1}(U) \in F_{\widetilde{X}}$. Because p is an open map, $p(p^{-1}(U))$ is open in X. Because p is onto $p(p^{-1}(U)) = U$. Thus, $U \in F_X$.

$F_X = \{U \subseteq X \mid p^{-1}(U) \in F_X\}$, so p is a quotient map and X is a quotient space. $\square$

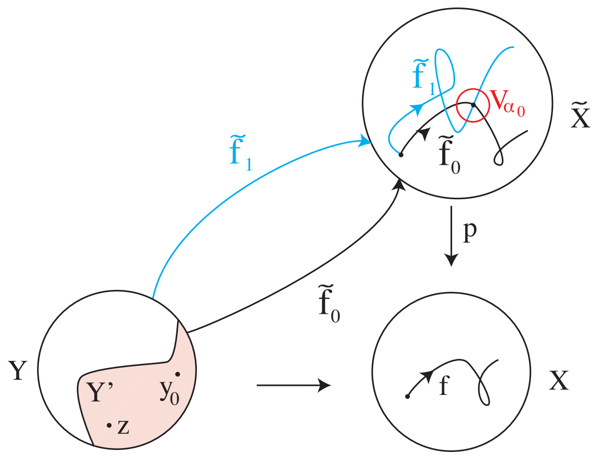
17.1. Lifts.

DEFINITION. Let $p : \widetilde{X} \to X$ be a covering map and $f : Y \to X$ be continuous. We define a **lift** of f to be any continuous function $\widetilde{f} : Y \to X$ such that $p \circ \widetilde{f} = f$.

EXAMPLE. Let $\widetilde{X} = \mathbb{R}$, $X = S^1$ and $p(x) = (\cos(2\pi x), \sin(2\pi x))$. Let $f : I \to S^1$ by $f(x) = (\cos(\pi x), \sin(\pi x))$. Define $\widetilde{f} : I \to \mathbb{R}$ by $f(x) = \frac{x}{2}$. Then $\widetilde{f}$ is a lift of f.



Lemma. ***Uniqueness of Lifts.*** *Let $p : \widetilde{X} \to X$ be a covering map and $f : Y \to X$ be continuous and Y be connected. Let $\widetilde{f}_0$ and $\widetilde{f}_1$ be lifts of f. Suppose there exists $y_0 \in Y$ such that $\widetilde{f}_0(y_0) = \widetilde{f}_1(y_0)$. Then $\widetilde{f}_0 = \widetilde{f}_1$.*



Proof. Let $Y' = \{y \in Y \mid \widetilde{f}_0(y) = \widetilde{f}_1(y)\}$, then $y_0 \in Y'$. We want to show that $Y' = Y$, we will accomplish this by showing Y' is clopen in Y (which is connected).

Open: Let $y \in Y'$. Then there exists an evenly covered open set V containing $f(y)$. Thus $p^{-1}(V) = \bigcup\limits_{\alpha \in A} V_\alpha$ such that V_α's are disjoint and open and $p \mid V_\alpha : V_\alpha \to V$ is a homeomorphism.

Let $q = \widetilde{f}_0(y) = \widetilde{f}_1(y)$. There exists an $\alpha_0 \in A$ such that $q \in V_{\alpha_0}$. Now $\widetilde{f}_0^{-1}(V_{\alpha_0})$ and $\widetilde{f}_1^{-1}(V_{\alpha_0})$ are open in Y, and $y \in \widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_0})$. We claim that $\widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_0}) \subseteq Y'$.

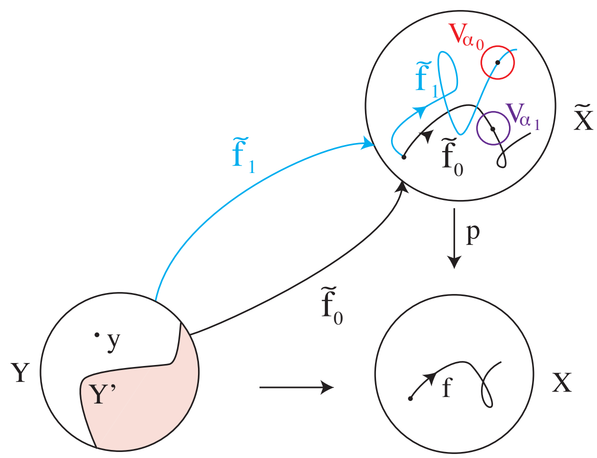
Let $z \in \widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_0})$. Then $\widetilde{f}_0(z) \in V_{\alpha_0}$ and $\widetilde{f}_1(z) \in V_{\alpha_0}$. Since $\widetilde{f}_0$ and $\widetilde{f}_1$ are lifts of f, we have $p \circ \widetilde{f}_0(z) = f(z)$ and $p \circ \widetilde{f}_1(z) = f(z)$. Since $p|V_{\alpha_0}$

is 1-1 (because $p|V_{\alpha_0}$ is a homeomorphism) and $\widetilde{f}_0(z) \in V_{\alpha_0}$ and $\widetilde{f}_1(z) \in V_{\alpha_0}$, $\widetilde{f}_0(z) = \widetilde{f}_1(z)$. Thus, $z \in Y'$.

Hence, $\widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_0})$ is an open subset of Y' containing y, so Y' is open.

Closed: To show Y' is closed, we will show $Y - Y'$ is open. Let $y \in Y - Y'$. Then there exists an evenly covered open set V containing $f(y)$. Hence $p^{-1}(V) = \bigcup_{\alpha \in A} V_\alpha$ such that the V_α's are disjoint and open and $p \mid V_\alpha : V_\alpha \to V$ is a homeomorphism.

Note that $\widetilde{f}_0(y) \neq \widetilde{f}_1(y)$, and there exists $\alpha_0, \alpha_1 \in A$ such that $\widetilde{f}_0(y) \in V_{\alpha_0}$ and $\widetilde{f}_1(y) \in V_{\alpha_1}$. Now the set $\widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_1})$ is open and contains y. We claim that $\widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_1}) \subseteq Y - Y'$.



Let $z \in \widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_1})$. Hence $\widetilde{f}_0(z) \in V_{\alpha_0}$ and $\widetilde{f}_1(z) \in V_{\alpha_1}$. We now want to show that $\widetilde{f}_0(z) \neq \widetilde{f}_1(z)$ by showing $\alpha_0 \neq \alpha_1$. Recall that $p \mid V_{\alpha_0}$ is $1-1$ and $p \circ \widetilde{f}_0(y) = f(y) = p \circ \widetilde{f}_1(y)$. Since $\widetilde{f}_0(y) \neq \widetilde{f}_1(y)$, it follows that $\alpha_0 \neq \alpha_1$. Therefore, $V_{\alpha_0} \cap V_{\alpha_1} = \emptyset$.

Therefore $\widetilde{f}_0(z) \neq \widetilde{f}_1(z)$, implying that $z \in Y - Y'$. This implies y is contained in the open set $\widetilde{f}_0^{-1}(V_{\alpha_0}) \cap \widetilde{f}_1^{-1}(V_{\alpha_1}) \subseteq Y - Y'$, making $Y - Y'$ open.

Therefore Y' is clopen in Y. Because Y' is non-empty and Y is connected, Y' must be all of Y. By the definition of Y', $\widetilde{f}_0 = \widetilde{f}_1$. $\square$

How did we use $\widetilde{f}_0(y_0) = \widetilde{f}_1(y_1)$ in our proof?

18. More ***Fun***damental Groups

Now we are going to assume the HPLT and the Monodromy Theorem and use them to obtain results about the fundamental group. We'll prove these important results afterwards.

Note that in this section our results will be of mixed flavors

Our goal in this section is to determine the fundamental group of S^1 and a couple other spaces which are not simply connected. But we need just a teensy bit more machinery.

Definition. Let $p : \mathbb{R} \to S^1$ be the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Let $x_0 = (1, 0)$, let f be a loop in S^1 with base point x_0. Define the **degree** of f, denoted by $deg(f)$, as $\widetilde{f}(1)$ where $\widetilde{f}$ is the unique lift of f starting at 0.

Note that we know such a lift exists by the HPLT, and we saw by the lemma above that the lift is unique. Hence $deg(f)$ exists and is well defined. Observe that $p^{-1}(\{x_0\}) = \mathbb{Z}$.

Now we have the theorem we have all been waiting for.

Theorem. *Let $x_0 \in S^1$. Then $\pi_1(S^1, x_0) \cong \mathbb{Z}$.*

Proof. We assume WLOG that $x_0 = (1, 0)$ since S^1 is path connected. We will use the covering map $p\colon \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$, and the degree function defined above. Define the map $\varphi\colon \pi_1(S^1, x_0) \to \mathbb{Z}$ by $\varphi([f]) = deg(f)$ for every loop f in S^1 based at x_0. Our aim is to show that φ is an isomorphism.

Well-Defined: Suppose $[f] = [g]$. Let $\widetilde{f}$ and $\widetilde{g}$ be the unique lifts of f and g respectively based at 0 (which exist by HPLT). Since $f \sim g$ are homotopic loops in S^1 based at x_0, by the Monodromy theorem we know $\widetilde{f}(1) = \widetilde{g}(1)$. In particular, $deg(f) = deg(g)$, and hence φ is well-defined.

1-1: Let $[f], [g] \in \pi_1(S^1, x_0)$ be such that $\varphi([f]) = \varphi([g])$. This means that $deg(f) = deg(g)$, and hence $\widetilde{f}(1) = \widetilde{g}(1)$. But $\mathbb{R}$ is simply connected, so since $\widetilde{f}$ and $\widetilde{g}$ agree on their endpoints, $\widetilde{f} \sim \widetilde{g}$. Hence there exists a path homotopy $\widetilde{F}\colon I \times I \to \mathbb{R}$ such that $\widetilde{F}(0, t) = \widetilde{f}$ and $\widetilde{F}(1, t) = \widetilde{g}$. Observe that $p \circ \widetilde{F}$ is continuous because it is a composition of continuous functions, and $p \circ \widetilde{F}(0, t) = p \circ \widetilde{f} = f$; $p \circ \widetilde{F}(1, t) = p \circ \widetilde{g} = g$. Finally, we know that for all $s \in I$ we have $\widetilde{F}(s, 0) = \widetilde{f}(1)$, so $p \circ \widetilde{F}(s, 0) = p \circ \widetilde{f}(1) = f(1) = x_0$, and the same holds for $t = 1$, so $p \circ \widetilde{F}$ is a path homotopy F which takes f to g, and thus $[f] = [g]$.

Onto: Let $n \in \mathbb{Z}$. Since $\mathbb{R}$ is path connected it contains a path $\widetilde{f}$ from 0 to n. Then $p \circ \widetilde{f}$ is a loop in S^1 based at x_0 and $deg(p \circ \widetilde{f}) = n$. Therefore $\varphi([p \circ \widetilde{f}]) = n$. This proves that φ is onto.

Observe that our proof that φ is a well defined bijection only uses the fact that the covering space $\mathbb{R}$ is simply connected. Keep this in mind for our next result.

Homo: Let $[f], [g] \in \pi_1(S^1, x_0)$. We want to show that $\varphi([f][g]) = \varphi([f]) + \varphi([g])$. Note that the right hand side is equal to $deg(f) + deg(g)$, while the left hand side is equal to $deg(f * g)$. So we want to show that $deg(f * g) = deg(f) + deg(g)$. Let $\widetilde{f * g}$ be the lift of $f * g$ beginning at 0. Note that $\widetilde{f} * \widetilde{g}$ is not defined because $\widetilde{f}(1) \neq \widetilde{(g)}(0)$.

We want to show that $\widetilde{f * g}(1) = \widetilde{f}(1) + \widetilde{g}(1)$. Let $m = \widetilde{f}(1), n = \widetilde{g}(1)$, so $\widetilde{f}(1) + \widetilde{g}(1) = m + n$. So we want to show that $\widetilde{f * g}(1) = m + n$. Note that $\widetilde{f}(1) \neq \widetilde{g}(0)$ so $\widetilde{f} * \widetilde{g}$ is not defined. Thus we define a function $h \colon I \to \mathbb{R}$ by:

$$h(s) = \begin{cases} \widetilde{f}(2s) & s \in [0, \frac{1}{2}] \\ \widetilde{g}(2s-1) + m & s \in [\frac{1}{2}, 1] \end{cases}$$

Since $\widetilde{f}$ and $\widetilde{g}$ are continuous, and when $s = \frac{1}{2}$ we have $\widetilde{f}(1) = m$, and $\widetilde{g}(0) + m = 0 + m = m$, by the Pasting Lemma h is continuous. Also, $h(0) = 0$ and $h(1) = \widetilde{g}(1) + m = n + m$. Thus h is a path in $\mathbb{R}$ from 0 to $m + n$. We show as follows that $p \circ h = f * g$.

$$p \circ h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ p(\widetilde{g}(2s-1) + m) & s \in [\frac{1}{2}, 1] \end{cases}$$

Now observe that

$$\begin{aligned} p(\widetilde{g}(2s-1) + m) &= (\cos 2\pi(\widetilde{g}(2s-1) + m), \sin 2\pi(\widetilde{g}(2s-1) + m)) \\ &= (\cos 2\pi\widetilde{g}(2s-1), \sin 2\pi\widetilde{g}(2s-1)) \\ &= p(\widetilde{g}(2s-1)) \\ &= g(2s-1) \end{aligned}$$

Thus we see that $p \circ h = f * g$. Hence h is the unique lift of $f * g$ which begins at 0. It follows that $h = \widetilde{f * g}$, and hence $h(1) = m + n = \widetilde{f}(1) + \widetilde{g}(1)$. Thus

$$deg(f * g) = deg(f) + deg(g) \Rightarrow \varphi([f][g]) = \varphi([f]) + \varphi([g])$$

So we conclude that φ is an isomorphism, and hence $\mathbb{Z} \cong \pi_1(S^1, x_0)$. $\square$

We now rejoice in the fact that we have seen our first non-trivial fundamental group.

Here is a useful theorem about the size of the fundamental group of a space.

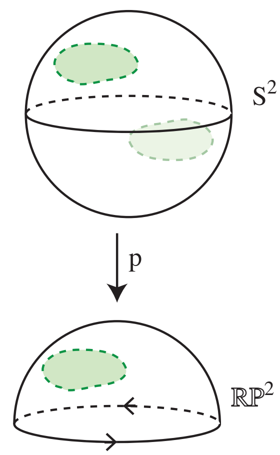
THEOREM. *Let $x_0 \in X$ and $p \colon \widetilde{X} \to X$ be a covering map. If $\widetilde{X}$ is simply connected, then there exists a bijection from $\pi_1(X, x_0)$ to $p^{-1}(\{x_0\})$.*

PROOF. Let $y_0 \in p^{-1}(\{x_0\})$, and let $\varphi \colon \pi_1(X, x_0) \to p^{-1}(\{x_0\})$ by $\varphi([f]) = \widetilde{f}(1)$, where $\widetilde{f}$ is the unique lift of f originating at y_0. Recall that we only needed simple connectedness of the covering space in the proof of the above theorem that φ was a bijection. So we can use an identical argument here. $\square$

Now the fun begins.

THEOREM. $\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}_2$ *(the group consisting of* $\{0, 1\}$ *with mod 2 arithmetic written by algebraists as* $\mathbb{Z}/2\mathbb{Z}$*).*

PROOF. Define an equivalence relation on S^2 as $x \sim y$ iff $x = \pm y$. We can see that the quotient map is $p\colon S^2 \to \mathbb{RP}^2$. Consider any open disk in $\mathbb{RP}^2$. Then its pre-image will be a pair of disjoint open disks such that p restricted to one of these disks will be a homeomorphism (see figure). Thus such an open disk in $\mathbb{RP}^2$ is evenly covered.



Hence p is a covering map. Let $x_0 \in \mathbb{RP}^2$. By the above theorem, there is a bijection from $\pi_1(\mathbb{RP}^2, x_0)$ to $p^{-1}(\{x_0\})$ (since S^2 is simply connected by a homework problem), and we know that $p^{-1}(\{x_0\})$ has precisely two elements, so $\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}_2$ (the only group with precisely two elements). $\square$

Skip this if we didn't do homotopy equivalent implies isomorphic fundamental groups

THEOREM. $\mathbb{R}^2 \not\cong \mathbb{R}^3$

PROOF. First, recall some results:

- $\mathbb{R}^{n+1} - \{p\}$ is homotopy equivalent to S^n for all $n \geq 1$ (by Homework 10)
- $\pi_1(S^2, x_0)$ is trivial (by Homework 10)
- $\pi_1(S^1, x_0) \cong \mathbb{Z}$

Suppose there exists some homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^3$. Let $p \in \mathbb{R}^2$ and $h(p) = q \in \mathbb{R}^3$. Therefore $f = h|_{\mathbb{R}^2 - \{p\}} : \mathbb{R}^2 - \{p\} \to \mathbb{R}^3 - \{q\}$ is a homeomorphism. Also, since we have shown that homotopy equivalent spaces have isomorphic fundamental groups, we have the following results.

- $\pi_1(\mathbb{R}^2 - \{p\}, x_0) \cong \pi_1(S^1, y_0) \cong \mathbb{Z}$

- $\pi_1(\mathbb{R}^3, -\{q\}, z_0) \cong \pi_1(S^2, w_0) \cong \{1\}$

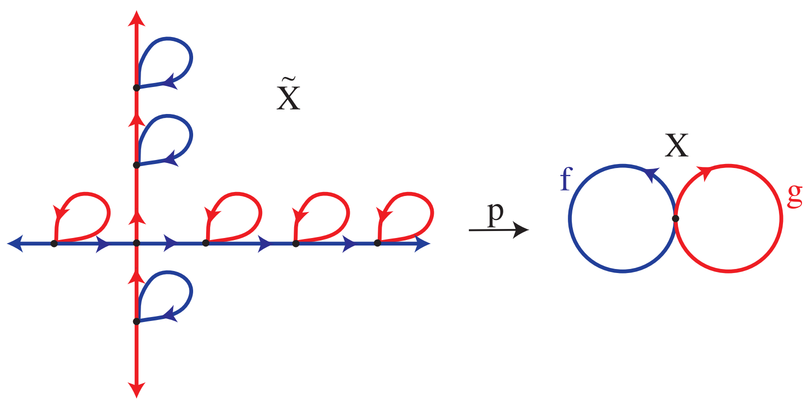
But since f is a homeomorphism, $f_* : \pi_1(\mathbb{R}^2 - \{p\}, x_0) \to \pi_1(\mathbb{R}^3, -\{q\}, z_0)$ is an isomorphism. However, $\mathbb{Z} \ncong \{1\}$, so this is a contradiction.

Therefore $\mathbb{R}^2 \ncong \mathbb{R}^3$, as desired. $\square$

Theorem. *Let $X = S^1 \vee S^1$ with wedge point x_0. Then $\pi_1(X, x_0)$ is not abelian (i.e., non-commutative).*

Proof. Let Y denote the union of the two axes in $\mathbb{R}^2$, and let $\widetilde{X}$ be Y with a copy of S^1 wedged at every point of the form $(z, 0)$ and $(0, z)$ with $z \in \mathbb{Z} - \{0\}$ (see the figure below).

Define $p : \widetilde{X} \to X$ such that each blue circle and each blue segment of unit length go to the blue circle, and each red circle and each red segment of unit length go to the red circle. We can check on the picture to see that every point in S^1 has an evenly covered open set around it. Hence p is indeed a covering map.



Let f be a single loop around the blue circle and let g be a single loop around the red circle. Now lift $f * g$ and $g * f$ beginning at the origin in $\widetilde{X}$. The construction of $\widetilde{X}$ gives us that $\widetilde{f * g}(1) = (1, 0)$ and $\widetilde{g * f}(1) = (0, 1)$. Since these are distinct lifts with the same starting point, by the Uniqueness of Lifts Theorem we must have $f * g \not\simeq g * f$. Thus $\pi_1(X, x_0)$ is not abelian, as desired.$\square$

Thus $\pi_1(X, x_0)$ is a non-trivial group which is different from those we've seen so far.

19. Proofs of HPLT and Monodromy

Lemma (Lebesgue Number Lemma). *Let X be a compact metric space and let Ω be an open cover of X. Then $\exists\ r > 0$ such that $\forall\ A \subseteq X$ with $lub\{d(p, q) | p, q \in A\} < r$, A is contained in a single element of Ω.*

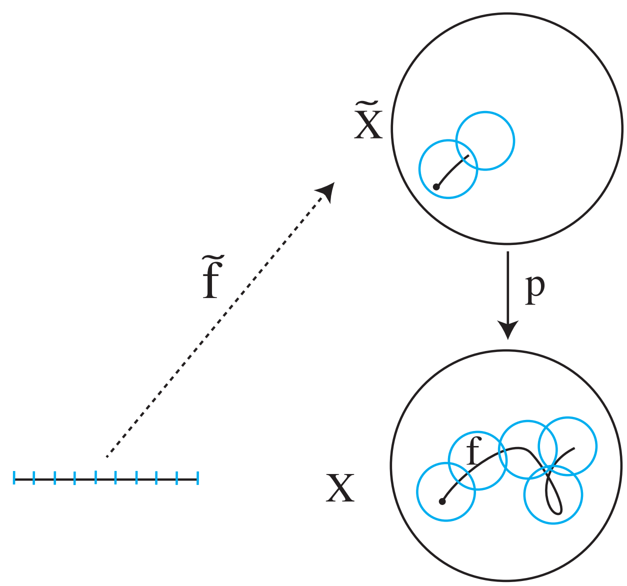
(r is said to be a Lebesgue Number for Ω)

We proved this lemma in the homework. We will now use it to prove the existence of lifts.

Theorem (Very Important Homotopy Path Lifting Theorem). *Let $p : \widetilde{X} \to X$ be a covering map. Then,*

(1) *Given a path f in X and $a \in \widetilde{X}$ such that $p(a) = f(0)$, then $\exists!$ (exists unique) path $\widetilde{f}$ in $\widetilde{X}$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(0) = a$.*

(2) *Given a continuous map $F : I \times I \to X$ and $a \in \widetilde{X}$ with $p(a) = F(0,0)$, $\exists!$ continuous map $\widetilde{F}\colon I \times I \to \widetilde{X}$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(0,0) = a$.*



Proof. 1) $\forall\, x \in f(I)$, $\exists\, V_x$ an evenly covered open set containing x. $\forall\, x \in f(I)$, $f^{-1}(V_x)$ is open in I, so $\{f^{-1}(V_x) | x \in f(I)\}$ is an open cover of I. So, $\exists$ Lebesgue number r for this cover. Now $\exists\, n \in \mathbb{N}$ such that $\frac{1}{n} < r$. Thus $\forall\, k \leq n$, $[\frac{k-1}{n}, \frac{k}{n}]$ is contained entirely in some $f^{-1}(V_x)$. So, $\exists\, \{V_1, V_2, ..., V_n\} \subseteq \{V_x\}$ such that $\forall\, k \leq n$, $f([\frac{k-1}{n}, \frac{k}{n}]) \subseteq V_k$.

First, V_1 is evenly covered and $f(0) \in V_1$, so $a \in p^{-1}(V_1) = \bigcup_{\alpha \in A_1} V_\alpha$. So, $\exists\, \alpha_1 \in A_1$ such that $a \in V_{\alpha_1}$. $p|V_{\alpha_1}\colon\ V_{\alpha_1} \to V_1$ is a homeomorphism. So $\forall\, s \in [0, \frac{1}{n}]$, define $\widetilde{f}(s) = (p|V_{\alpha_1})^{-1} f(s)$. Note that $\widetilde{f}\colon\ [0, \frac{1}{n}] \to \widetilde{X}$ continuous because $(p|V_{\alpha_1})^{-1}$ is a homeomorphism. Now note that, as above, $p^{-1}(V_2) = \bigcup_{\alpha \in A_2} V_\alpha$. $f(\frac{1}{n}) \in V_2$ by definition. $\exists\, \alpha_2 \in A_2$ such that $\widetilde{f}(\frac{1}{n}) \in V_{\alpha_2}$. So, as above, define $\widetilde{f}\colon\ [\frac{1}{n}, \frac{2}{n}] \to \widetilde{X}$ by $\widetilde{f}(s) = (p|V_{\alpha_2})^{-1} f(s)$. $\widetilde{f}\colon\ [0, \frac{2}{n}] \to \widetilde{X}$ is therefore continuous by Pasting Lemma. Continue this process to define $\widetilde{f}$. Furthermore, $\widetilde{f}$ is unique by the Uniqueness Lemma which we already proved.

2) $\forall x \in F(I \times I)$ $\exists$ evenly covered open set V_x. $\{F^{-1}(V_x)\}$ is an open cover of $I \times I$, so it has a Lesbegue number r. $\exists\, n > \frac{\sqrt{2}}{r}$. $\forall\, i \leq n$, let $A_i = [\frac{i-1}{n}, \frac{i}{n}]$, $B_i = [\frac{i-1}{n}, \frac{i}{n}]$. $\forall\, i, j$,

$F(A_i \times B_j) \subseteq V_{ij}$ for some $V_{ij} \in \{V_x\}$. By Part 1, we can lift $F|(I \times \{0\} \cup \{0\} \times I)$ to $\widetilde{F}$: $(I \times \{0\} \cup \{0\} \times I) \to \widetilde{X}$ such that $\widetilde{F}(0,0) = a \in \widetilde{X}$.

Begin by observing that V_{11} is evenly covered by hypothesis, and $F(I_1 \times J_1) \subseteq V_{11}$. This means that $p^{-1}(V_{11}) = \bigcup_{\alpha \in A_{11}} V_\alpha$ where the V_α are disjoint open sets and A_{11} is some index set. Since $F(0,0) \in V_{11}$, and since $\widetilde{F}(0,0) = a$, it follows that there exists some $\alpha_{11} \in A_{11}$ such that $a \in V_{\alpha_{11}}$.

Now we worry that this choice of $V_{\alpha_{11}}$ will agree with how we defined $\widetilde{F}$ on the set L. Worry not! For L is connected, and $L \cap (I_1 \times J_1)$ is connected, and since $\widetilde{F}$ is continuous, $\widetilde{F}(L \cap (I_1 \times J_1))$ is connected. Since the V_α are open and disjoint, we may therefore conclude that $\widetilde{F}(L \cap (I_1 \times J_1)) \subseteq V_{\alpha_{11}}$ (otherwise it would be disconnected).

Since V_{11} was evenly covered, we know that $p \mid V_{\alpha_{11}}$ is a homeomorphism, so we may define $\widetilde{F}\colon I_1 \times J_1 \to \widetilde{X}$ by:

$$\widetilde{F}(s,t) = (p \mid V_{\alpha_{11}})^{-1} \circ F(s,t)$$

This is a composition of continuous functions, so is continuous. Furthermore, $\widetilde{F}\colon L \cup (I_1 \times J_1) \to \widetilde{X}$ is continuous since we showed that $\widetilde{F}(L \cap (I_1 \times J_1)) \subseteq V_{\alpha_{11}}$, so we apply the Pasting Lemma.

Now we want to extend $\widetilde{F}$ to the rest of $I \times I$, and so we move to $I_2 \times J_1$. Here we have an analogous situation as before: We want to choose the appropriate V_α associated with V_{21} so that our extension of $\widetilde{F}$ agrees with what we had previously. But again, $\widetilde{F}((I_2 \times J_1) \cap (L \cup (I_1 \times J_1)))$ is connected, so following the argument from above there will be an appropriate choice of V_α to make it "work". So we inductively define $\widetilde{F}\colon I \times I \to \widetilde{X}$ such that it is continuous as before, and $p \circ \widetilde{F} = F$ and $\widetilde{F}(0,0) = a$. That $\widetilde{F}$ is unique follows from our Uniqueness of Lifts Lemma above.

We iterate this argument for each tile $I_i \times J_j$, and so inductively define a unique lift of F: $\widetilde{F}\colon I \times I \to \widetilde{X}$ such that $\widetilde{F}(0,0) = a$. $\square$

The natural intuition is that our new function $\widetilde{F}$ is a path homotopy when F is a path homotopy. This intuition provides a delightful segue to the next theorem:

Theorem (Monodromy Theorem). *Let $p : \widetilde{X} \to X$ be a covering map, and let $a \in \widetilde{X}$. Let $x_1, x_2 \in X$. Suppose that $p(a) = x_1$, and that f, g are paths in X from x_1 to x_2 such that $f \sim g$. Let $\widetilde{f}, \widetilde{g}$ be the unique lifts of f, g beginning at a. Then $\widetilde{f}(1) = \widetilde{g}(1)$ and $\widetilde{f} \sim \widetilde{g}$.*

Before beginning the proof, we observe with relish the etymology of monodromy. Mono being the prefix for one, and dromy being some sort of Greek for a race track. So in a sense monodromy means one path.

Proof. $f \sim g$ means that there exists a path homotopy $F\colon I \times I \to X$, and so by the previous theorem there exists a unique lifting of F, whose name is $\widetilde{F}\colon I \times I \to \widetilde{X}$, and $\widetilde{F}$

has the property that $\widetilde{F}(0,0) = a$ and $p \circ \widetilde{F} = F$. Now $\widetilde{f}, \widetilde{g}$ are lifts of f, g respectively. Consider $\widetilde{F} \mid (I \times \{0\})$. This is a path in $\widetilde{X}$ from a to $\widetilde{F}(1,0)$. Observe that:

$$p \circ \widetilde{F} \mid (I \times \{0\}) = F \mid (I \times \{0\}) = f$$

since F was a path homotopy, and on the other hand:

$$p \circ \widetilde{F} \mid (I \times \{1\}) = F \mid (I \times \{1\}) = g$$

The first observation allows us to conclude that $\widetilde{F} \mid (I \times \{0\})$ is a lift of f beginning at a. By the uniqueness of lifts, we conclude that $\widetilde{F} \mid (I \times \{0\}) = \widetilde{f}$. We want to say the same for $\widetilde{F} \mid (I \times \{1\})$, but we do not know that $\widetilde{F}(0,1) = a$, so we cannot immediately conclude that this is equal to $\widetilde{g}$ since it could possibly be a lift of g originating at some other point.

We claim: $\widetilde{F} \mid (\{0\} \times I) = a$. To see that this is the case, we know:

$$p \circ \widetilde{F} \mid (\{0\} \times I) = F \mid (\{0\} \times I) = x_1.$$

which implies that

$$\widetilde{F} \mid (\{0\} \times I) \subseteq p^{-1}(F \mid (\{0\} \times I) = p^{-1}(\{x_1\})$$

Now we know that since p is a covering map, $p^{-1}(\{x_1\})$ has the discrete topology. Also, $\widetilde{F}(\{0\} \times I)$ is connected, so must contain only a single point of $p^{-1}(\{x_1\})$. Certainly $a \in \widetilde{F}(\{0\} \times I)$, so $a = \widetilde{F}(\{0\} \times I)$ as desired.

The previous consideration tells us that $\widetilde{F} \mid (I \times \{1\}) = \widetilde{g}$, since the left hand side is a lift of g originating at a, and by the uniqueness of lifts this must be $\widetilde{g}$. To finish off proving that $\widetilde{F}$ is a path homotopy, we need to show that the endpoints are constant as well. That is, we want to show that $\widetilde{F}(\{1\} \times I) = a'$ for some $a' \in \widetilde{X}$. But for this, the same argument as above applies, replacing every instance of x_1 with x_2. So we conclude that:

$$\widetilde{F}(1,0) = \widetilde{f}(1) = \widetilde{g}(1) = \widetilde{F}(1,1)$$

which was part of what we were trying to prove. All these considerations together tell us that $\widetilde{F}$ is a path homotopy between $\widetilde{f}$ and $\widetilde{g}$, so $\widetilde{f} \sim \widetilde{g}$ and we are done. $\square$

Note: Loops may lift to paths, but it follows from the Monodromy Theorem that trivial loops lift to loops (why?).

20. Products

THEOREM. *Let X and Y be path connected and $x_0 \in X$ and $y_0 \in Y$. Then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. Define $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\varphi([f]) = (p_*([f]_{X\times Y}), q_*([f]_{X\times Y})) = ([pf]_X, [qf]_Y)$.

Well-defined: Suppose $f \sim f'$ by a path homotopy $F : I \times I \to X \times Y$. Then $p \circ F$ and $q \circ F$ are path homotopies from $p \circ f$ to $p \circ f'$ and from $q \circ f$ to $q \circ f'$ respectively. So $([pf]_X, [qf]_Y) = ([pf']_X, [qf']_Y)$ as required.

Homomorphism: $\varphi([f][g]) = (p_*([f][g]), q_*([f][g])$. SInce p_* and q_* are homomorphisms, this is $(p_*([f])p_*([g])), q_*([f])q_*([g]))$. By definition of the product of groups this is

$$(p_*([f]), q_*([f])) \times (p_*([g], q_*([g])) = \varphi([f]) \times \varphi([g]).$$

1-1: Suppose that $\varphi([f]) = \varphi([g])$. Then $([pf]_X, [qf]_Y) = ([pg]_X, [qg]_Y)$. Hence $p\circ f \sim p\circ g$ and $q\circ f \sim q\circ g$. Hence there are homotopies $P : I\times I \to X$ from $p\circ f$ to $p\circ g$ and $Q : I\times I \to Y$ from $q \circ f$ to $q \circ g$. Define $H : I \times I \to X \times Y$ by $H(s, t) = (P(s, t), Q(s, t))$. Then H is continuous since F and G are. Also $H(s, 0) = (P(s, 0), Q(s, 0)) = (p(f(s)), q(f(s)) = f(s)$ and similarly $H(s, 1) = g(s)$. Thus $f \sim g$.

Onto: Let $[f] \in \pi_1(X, x_0)$ and $[g] \in \pi_1(Y, y_0)$. Define $h : I \to X\times Y$ by $h(s) = (f(s), g(s))$. Then h is continuous since both f and g are. Also $h(0) = (x_0, y_0) = h(1)$. So $[h] \in \pi_1(X \times Y, (x_0, y_0))$, and $\varphi([h]) = ([ph], [qh]) = ([f], [g])$. So φ is onto. □

Example: It follows from this theorem that $\pi_1(T^2, x_0) = \mathbb{Z} \times \mathbb{Z}$.