Symmetry **2014**, *6*, 189-209; doi:10.3390/sym6020189

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Topological Symmetry Groups of Small Complete Graphs

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Received: 12 February 2014; in revised form: 2 April 2014 / Accepted: 2 April 2014 / Published: 8 April 2014

Abstract: Topological symmetry groups were originally introduced to study the symmetries of non-rigid molecules, but have since been used to study the symmetries of any graph embedded in $\mathbb{R}^3$. In this paper, we determine for each complete graph K_n with $n \leq 6$, what groups can occur as topological symmetry groups or orientation preserving topological symmetry groups of some embedding of the graph in $\mathbb{R}^3$.

Keywords: topological symmetry groups; molecular symmetries; complete graphs; spatial graphs

Classification: MSC 57M15, 57M25, 05C10, 92E10

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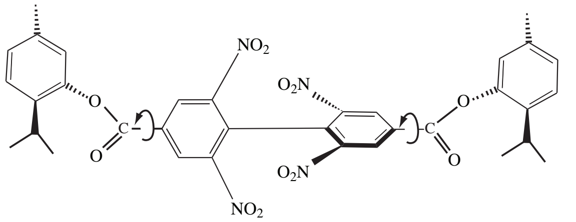
1. Introduction

Molecular symmetries are important in many areas of chemistry. Symmetry is used in interpreting results in crystallography, spectroscopy, and quantum chemistry, as well as in analyzing the electron structure of a molecule. Symmetry is also used in designing new pharmaceutical products. But what is meant by a "symmetry" depends on the rigidity of the molecule in question.

For rigid molecules, the group of rotations, reflections, and combinations of rotations and reflections, is an effective way of representing molecular symmetries. This group is known as the *point group*, of the molecule because it fixes a point of $\mathbb{R}^3$. However, some molecules can rotate around particular bonds, and large molecules can even be somewhat flexible. For example, supramolecular structures constructed through self-assembly may be somewhat conformationally flexible. Even relatively small molecules may contain rigid molecular subparts that rotate on hinges around particular bonds. For example, the left and right sides of the biphenyl derivative illustrated in Figure 1 rotate simultaneously, independent of the central part of the molecule. Because of these rotating pieces, this molecule is achiral though it cannot be rigidly superimposed on its mirror form. A detailed discussion of the achirality of this molecule can be found in [1].

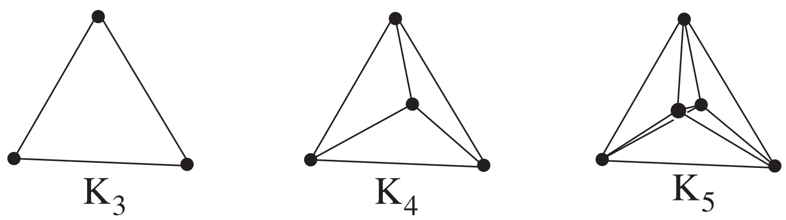
Figure 1. Because of its rotating subparts, this molecule is achiral.



In general, the amount of rigidity of a given molecule depends on its chemistry not just its geometry. Thus a purely mathematical definition of molecular symmetries that accurately reflects the behavior of all molecules is impossible. However, for non-rigid molecules, a topological approach to classifying symmetries including achirality can add important information beyond what is obtained from the point group. Such an approach could be useful to the study of supramolecular chirality, since structures constructed through self-assembly may be large and somewhat flexible or contain subparts that can rotate around covalent or non-convalent bonds.

The *topological symmetry group* was first introduced by Jon Simon in 1987 in order to classify the symmetries of non-rigid molecules [2]. By comparing the topological symmetry group and the orientation preserving topological symmetry group of a particular structure, one can see whether the structure is achiral and if so, understand how its achirality fits together with its other topological symmetries.

Figure 2. The graphs K_3, K_4, and K_5.



In this paper, we determine both the topological symmetry groups and the orientation preserving topological symmetry groups of structures whose underlying form is that of a complete graph with no more than six vertices. A *complete graph*, K_n, is defined to be a graph with n vertices which has an edge between every pair of vertices. In Figure 2 we illustrate embeddings of the complete graphs K_3, K_4, and K_5. The class of complete graphs is an interesting class to consider because the automorphism group of K_n is the symmetric group S_n, which is the largest automorphism group of any graph with n vertices. For small values of n, there exist molecules whose underlying topological structure has the form of K_n. For example, a tetrahedral supramolecular cluster has the underlying structure of the complete graph K_4.

If such a cluster contains a central atom which is bonded to the four corners of the tetrahedron, then the structure has the form of the complete graph K_5 (as illustrated on the right in Figure 2).

2. Background and Terminology

Though it may seem strange from the point of view of a chemist, the study of symmetries of embedded graphs is more convenient to carry out in the 3-dimensional sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ rather than in Euclidean 3-space, $\mathbb{R}^3$. In particular, in $\mathbb{R}^3$ every rigid motion is a rotation, reflection, translation, or a combination of these operations. Whereas, in S^3 glide rotations provide an additional type of rigid motion. While a topological approach to the study of symmetries does not require us to focus on rigid motions, for the purpose of illustration it is preferable to display rigid motions rather than isotopies whenever possible. Thus throughout the paper we work in S^3 rather than in $\mathbb{R}^3$.

Definition 1. *The* **topological symmetry group** *of a graph Γ embedded in S^3 is the subgroup of the automorphism group of the graph,* $\mathrm{Aut}(\Gamma)$*, induced by homeomorphisms of the pair* (S^3, Γ)*. The* **orientation preserving topological symmetry group**, $\mathrm{TSG}_+(\Gamma)$*, is the subgroup of* $\mathrm{Aut}(\Gamma)$ *induced by orientation preserving homeomorphisms of* (S^3, Γ).

It should be noted that for any homeomorphism h of (S^3, Γ), there is a homeomorphism g of (S^3, Γ) which fixes a point p not on Γ such that g and h induce the same automorphism on Γ. By choosing p to be the point at ∞, we can restrict g to a homeomorphism of $(\mathbb{R}^3, \Gamma)$. On the other hand if we start with an embedded graph Γ in $\mathbb{R}^3$ and a homeomorphism g of $(\mathbb{R}^3, \Gamma)$, we can consider Γ to be embedded in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and extend g to a homeomorphism of S^3 simply by fixing the point at ∞. It follows that the topological symmetry group of Γ in S^3 is the same as the topological symmetry group of Γ in $\mathbb{R}^3$. Thus we lose no information by working with graphs in S^3 rather than graphs in $\mathbb{R}^3$.

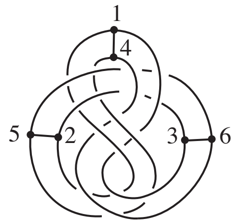
It was shown in [3] that the set of orientation preserving topological symmetry groups of 3-connected graphs embedded in S^3 is the same up to isomorphism as the set of finite subgroups of the group of orientation preserving diffeomorphisms of S^3, $\mathrm{Diff}_+(S^3)$. However, even for a 3-connected embedded graph Γ, the automorphisms in $\mathrm{TSG}(\Gamma)$ are not necessarily induced by finite order homeomorphisms of (S^3, Γ).

For example, consider the embedded 3-connected graph Γ illustrated in Figure 3. The automorphism (153426) is induced by a homeomorphism that slithers the graph along itself while interchanging the inner and outer knots in the graph. On the other hand, the automorphism (153426) cannot be induced by a finite order homeomorphism of S^3 because there is no order three homeomorphism of S^3 taking a figure eight knot to itself [4,5] and the embedded graph in Figure 3 cannot be pointwise fixed by a finite order homeomorphism of S^3 [6].

On the other hand, Flapan proved the following theorem which we will make use of later in the paper.

Finite Order Theorem. *[7] Let φ be a non-trivial automorphism of a* 3-connected *graph γ which is induced by a homeomorphism h of (S^3, Γ) for some embedding Γ of γ in S^3. Then for some embedding Γ' of γ in S^3, the automorphism φ is induced by a finite order homeomorphism, f of (S^3, Γ'), and f is orientation reversing if and only if h is orientation reversing.*

Figure 3. The topological symmetry group of this embedded graph is not induced by a finite group of homeomorphisms of S^3.



In the definition of the topological symmetry group, we start with a particular embedding Γ of a graph γ in S^3 and then determine the subgroup of the automorphism group of γ which is induced by homeomorphisms of (S^3, Γ). However, sometimes it is more convenient to consider all possible subgroups of the automorphism group of an abstract graph, and ask which of these subgroups can be the topological symmetry group or orientation preserving topological symmetry group of some embedding of the graph in S^3. The following definition gives us the terminology to talk about topological symmetry groups from this point of view.

Definition 2. *An automorphism f of an abstract graph, γ, is said to be* **realizable** *if there exists an embedding Γ of γ in S^3 such that f is induced by a homeomorphism of (S^3, Γ). A group G is said to be* **realizable for** *γ if there exists an embedding Γ of γ in S^3 such that $\mathrm{TSG}(\Gamma) \cong G$. If there exists an embedding Γ such that $\mathrm{TSG}_+(\Gamma) \cong G$, then we say G is* **positively realizable for** *γ.*

It is natural to ask whether every finite group is realizable. In fact, it was shown in [3] that the alternating group A_m is realizable for some graph if and only if $m \leq 5$. Furthermore, in [8] it was shown that for every closed, connected, orientable, irreducible 3-manifold M, there exists an alternating group A_m which is not isomorphic to the topological symmetry group of any graph embedded in M.

3. Topological Symmetry Groups of Compete Graphs

For the special class of complete graphs K_n embedded in S^3, Flapan, Naimi, and Tamvakis obtained the following result.

Complete Graph Theorem. *[9] A finite group H is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite subgroup of $\mathrm{SO}(3)$ or a subgroup of $\mathrm{D}_m \times \mathrm{D}_m$ for some odd m.*

This left open the question of what topological symmetry groups and orientation preserving topological symmetry groups are possible for embeddings of a particular complete graph K_n in S^3. For each $n > 6$, this question was answered for orientation preserving topological symmetry groups in the series of papers [10–13]. These papers make use of a result that for $n > 6$, only a few types of automorphisms of K_n are realizable [7]. There are no comparable results available for automorphisms of K_n when $n \leq 6$.

In the current paper, we determine which groups are realizable and which groups are positively realizable for each K_n with $n \leq 6$. This is the first family of graphs for which both the realizable and the positively realizable groups have been determined.

For $n \leq 3$, this question is easy to answer. In particular, since K_1 is a single vertex, the only realizable or positively realizable group is the trivial group. Since K_2 is a single edge, the only realizable or positively realizable group is $\mathbb{Z}_2$.

For $n = 3$, we know that $\mathrm{Aut}(K_3) \cong S_3 \cong \mathrm{D}_3$, and hence every realizable or positively realizable group for K_3 must be a subgroup of D_3. Note that for any embedding of K_3 in S^3, the graph can be "slithered" along itself to obtain an automorphism of order 3 which is induced by an orientation preserving homeomorphism. Thus the topological symmetry group and orientation preserving topological symmetry group of any embedding of K_3 will contain an element of order 3. Thus neither the trivial group nor $\mathbb{Z}_2$ is realizable or positively realizable for K_3. If Γ is a planar embedding of K_3 in S^3, then $\mathrm{TSG}(\Gamma) = \mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3$. Recall that the trefoil knot 3_1 is chiral while the knot 8_{17} is negative achiral and non-invertible. Thus if Γ is the knot 8_{17}, then no orientation preserving homeomorphism of (S^3, Γ) inverts Γ, but there is an orientation reversing homeomorphism of (S^3, Γ) which inverts Γ. Whereas, if Γ is the knot $8_{17} \# 3_1$, then there is no homeomorphism of (S^3, Γ) which inverts Γ. Table 1 summarizes our results for K_3.

Table 1. Realizable and positively realizable groups for K_3.

Embedding	**TSG(Γ)**	**TSG$_+$(Γ)**
Planar	D_3	D_3
8_{17}	D_3	$\mathbb{Z}_3$
8_{17} # 3_1	$\mathbb{Z}_3$	$\mathbb{Z}_3$

Determining which groups are realizable and positively realizable for K_4, K_5, and K_6 is the main point of this paper. In each case, we will first determine the positively realizable groups and then use the fact that either $\mathrm{TSG}_+(\Gamma) = \mathrm{TSG}(\Gamma)$ or $\mathrm{TSG}_+(\Gamma)$ is a normal subgroup of $\mathrm{TSG}(\Gamma)$ of index 2 to help us determine the realizable groups.

4. Topological Symmetry Groups of K_4

In addition to the Complete Graph Theorem given above, we will make use of the following results in our analysis of positively realizable groups for K_n with $n \geq 4$.

A_4 Theorem. *[11] A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that* $\mathrm{TSG}_+(\Gamma) \cong A_4$ *if and only if* $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.

A_5 Theorem. *[11] A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that* $\mathrm{TSG}_+(\Gamma) \cong A_5$ *if and only if* $m \equiv 0, 1, 5, 20 \pmod{60}$.

S_4 Theorem. *[11] A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that* $\mathrm{TSG}_+(\Gamma) \cong \mathrm{S}_4$ *if and only if* $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.

Subgroup Theorem. *[12] Let Γ be an embedding of a* 3-connected *graph γ in S^3 with an edge that is not pointwise fixed by any non-trivial element of* $\mathrm{TSG}_+(\Gamma)$. *Then every subgroup of* $\mathrm{TSG}_+(\Gamma)$ *is positively realizable for γ.*

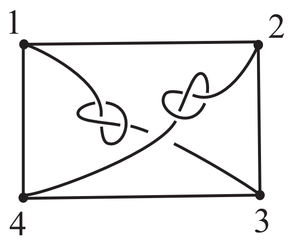
It was shown in [12] that adding a local knot to an edge of a 3-connected graph is well-defined and that any homeomorphism of S^3 taking the graph to itself must take an edge with a given knot to an edge with the same knot. Furthermore, any orientation preserving homeomorphism of S^3 taking the graph to itself must take an edge with a given non-invertible knot to an edge with the same knot oriented in the same way. Thus for $n > 3$, adding a distinct knot to each edge of an embedding of K_n in S^3 will create an embedding Δ where $\mathrm{TSG}(\Delta)$ and $\mathrm{TSG}_+(\Delta)$ are both trivial. Hence we do not include the trivial group in our list of realizable and positively realizable groups for K_n when $n > 3$.

Finally, observe that for $n > 3$, for a given embedding Γ of K_n we can add identical chiral knots (whose mirror image do not occur in Γ) to every edge of Γ to get an embedding Γ' such that $\mathrm{TSG}(\Gamma') = \mathrm{TSG}_+(\Gamma)$. Thus every group which is positively realizable for K_n is also realizable for K_n. We will use this observation in the rest of our analysis.

The following is a complete list of all the non-trivial subgroups of $\mathrm{Aut}(K_4) \cong \mathrm{S}_4$ up to isomorphism: S_4, A_4, D_4, D_3, D_2, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_2$.

We will show that all of these groups are positively realizable, and hence all of the groups will also be realizable. First consider the embedding Γ of K_4 illustrated in Figure 4. The square $\overline{1234}$ must go to itself under any homeomorphism of (S^3, Γ). Hence $\mathrm{TSG}_+(\Gamma)$ is a subgroup of D_4. In order to obtain the automorphism (1234), we rotate the square $\overline{1234}$ clockwise by $90°$ and pull $\overline{24}$ under $\overline{13}$. We can obtain the transposition (13) by first rotating the figure by $180°$ about the axis which contains vertices 2 and 4 and then pulling $\overline{13}$ under $\overline{24}$. Thus $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_4$. Furthermore, since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups $\mathbb{Z}_4$, D_2 and $\mathbb{Z}_2$ are each positively realizable for K_4.

Figure 4. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_4$.



Next, consider the embedding, Γ of K_4 illustrated in Figure 5. All homeomorphisms of (S^3, Γ) fix vertex 4. Hence $\mathrm{TSG}_+(\Gamma)$ is a subgroup of D_3. The automorphism (123) is induced by a rotation, and the automorphism (12) is induced by turning the figure upside down and then pushing vertex 4 back up through the centre of $\overline{123}$. Thus $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, by the Subgroup Theorem, the group $\mathbb{Z}_3$ is also positively realizable for K_4.

Thus every subgroup of $\mathrm{Aut}(K_4)$ is positively realizable. Now by adding appropriate equivalent chiral knots to each edge, all subgroups of $\mathrm{Aut}(K_4)$ are also realizable. We summarize our results for K_4 in Table 2.

Figure 5. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3$.

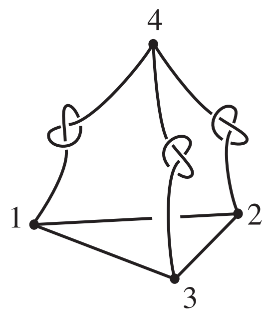


Table 2. Non-trivial realizable and positively realizable groups for K_4.

Subgroup	Realizable/Positively Realizable	Reason
S_4	Yes	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_4	Yes	By Figure 4
D_3	Yes	By Figure 5
D_2	Yes	By Subgroup Theorem
$\mathbb{Z}_4$	Yes	By Subgroup Theorem
$\mathbb{Z}_3$	Yes	By Subgroup Theorem
$\mathbb{Z}_2$	Yes	By Subgroup Theorem

5. Topological Symmetry Groups of K_5

The following is a complete list of all the non-trivial subgroups of $\mathrm{Aut}(K_5) \cong \mathrm{S}_5$:

S_5, A_5, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, D_6, D_5, D_4, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ (see [14] and [15]).

The lemma below follows immediately from the Finite Order Theorem [7] (stated in the introduction) together with Smith Theory [6].

Lemma 1. *Let $n > 3$ and let φ be a non-trivial automorphism of K_n which is induced by a homeomorphism h of (S^3, Γ) for some embedding Γ of K_n in S^3. If h is orientation reversing, then φ fixes at most 4 vertices. If h is orientation preserving, then φ fixes at most 3 vertices, and if φ has even order, then φ fixes at most 2 vertices.*

We now prove the following lemma.

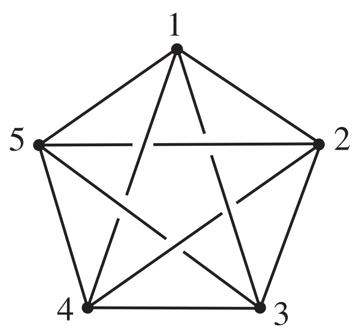
Lemma 2. *Let $n > 3$ and let Γ be an embedding of K_n in S^3 such that $\mathrm{TSG}_+(\Gamma)$ contains an element φ of even order $m > 2$. Then φ does not fix any vertex or interchange any pair of vertices.*

Proof. By the Finite Order Theorem, K_n can be re-embedded as Γ' so that φ is induced on Γ' by a finite order orientation preserving homeomorphism h of (S^3, Γ'). Suppose that φ fixes a vertex or interchanges a pair of vertices of Γ'. Then $\mathrm{fix}(h)$ is non-empty, and hence by Smith Theory, $\mathrm{fix}(h) \cong S^1$. Let $r = m/2$. Then h^r induces an involution on the vertices of Γ', and this involution can be written as a product $(a_1b_1)\cdots(a_qb_q)$ of disjoint transpositions of vertices. Now for each i, h^r fixes a point on the edge $\overline{a_ib_i}$. But $\mathrm{fix}(h^r)$ contains $\mathrm{fix}(h)$ and thus by Smith Theory $\mathrm{fix}(h^r) = \mathrm{fix}(h)$. Hence h fixes a point on each edge $\overline{a_ib_i}$. Thus h induces also $(a_1b_1)\cdots(a_qb_q)$ on the vertices of Γ'. But this contradicts the hypothesis that the order of φ is $m > 2$. □

By Lemma 2, there is no embedding of K_5 in S^3 such that $\mathrm{TSG}_+(\Gamma)$ contains an element of order 4 or of order 6. It follows that $\mathrm{TSG}_+(\Gamma)$ cannot be D_6, $\mathbb{Z}_6$, D_4 or $\mathbb{Z}_4$.

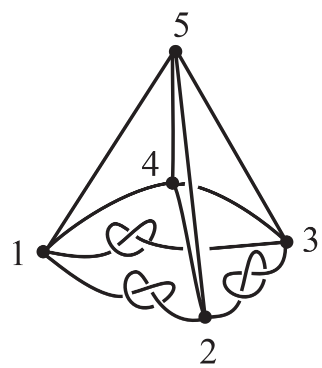
Consider the embedding Γ of K_5 illustrated in Figure 6. The knotted cycle $\overline{13524}$ must be setwise invariant under every homeomorphism of Γ. Thus $\mathrm{TSG}_+(\Gamma) \leq D_5$. The automorphism (12345) is induced by rotating Γ, and $(25)(34)$ is induced by turning the graph over. Hence $\mathrm{TSG}_+(\Gamma) = \langle(12345), (25)(34)\rangle \cong \mathrm{D}_5$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups $\mathbb{Z}_5$ and $\mathbb{Z}_2$ are also positively realizable for K_5.

Figure 6. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_5$.



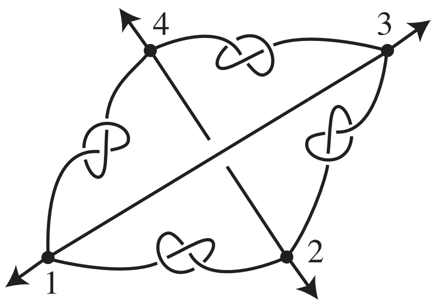
Next consider the embedding Γ of K_5 illustrated in Figure 7. The triangle $\overline{123}$ must go to itself under any homeomorphism. Also by Lemma 1, any orientation preserving homeomorphism which fixes vertices 1, 2, and 3 induces a trivial automorphism on K_5. Thus $\mathrm{TSG}_+(\Gamma) \leq \mathrm{D}_3$. The automorphism (123) is induced by a rotation. Also the automorphism $(45)(12)$ is induced by pulling vertex 4 down through the centre of triangle $\overline{123}$ while pulling vertex 5 into the centre of the figure then rotating by 180° about the line through vertex 3 and the midpoint of the edge $\overline{12}$. Thus $\mathrm{TSG}_+(\Gamma) = \langle(123), (45)(12)\rangle \cong \mathrm{D}_3$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, by the Subgroup Theorem, the group $\mathbb{Z}_3$ is positively realizable for K_5.

Figure 7. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3$.



Lastly, consider the embedding Γ of K_5 illustrated in Figure 8 with vertex 5 at infinity. The square $\overline{1234}$ must go to itself under any homeomorphism. Hence $\mathrm{TSG}_+(\Gamma) \leq \mathrm{D}_4$. The automorphism $(13)(24)$ is induced by rotating the square by $180°$. By turning over the figure we obtain $(12)(34)$. By Lemma 2, $\mathrm{TSG}_+(\Gamma)$ cannot contain an element of order 4. Thus $\mathrm{TSG}_+(\Gamma) = \langle(13)(24), (12)(34)\rangle \cong \mathrm{D}_2$.

Figure 8. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_2$.



We summarize our results on positive realizability for K_5 in Table 3.

Table 3. Non-trivial positively realizable groups for K_5.

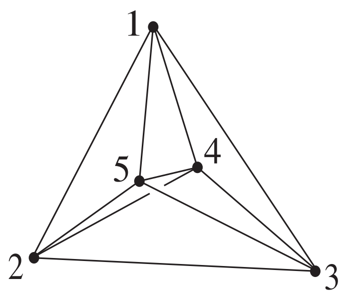
Subgroup	**Positively Realizable**	**Reason**
S_5	No	By Complete Graph Theorem
A_5	Yes	By A_5 Theorem
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	No	By Complete Graph Theorem
S_4	No	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_6	No	By Lemma 2
D_5	Yes	By Figure 6
D_4	No	By Lemma 2
D_3	Yes	By Figure 7
D_2	Yes	By Figure 8
$\mathbb{Z}_6$	No	By Lemma 2
$\mathbb{Z}_5$	Yes	By Subgroup Theorem
$\mathbb{Z}_4$	No	By Lemma 2
$\mathbb{Z}_3$	Yes	By Subgroup Theorem
$\mathbb{Z}_2$	Yes	By Subgroup Theorem

Again by adding appropriate equivalent chiral knots to each edge, all of the positively realizable groups for K_5 are also realizable. Thus we only need to determine realizability for the groups S_5, S_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, D_6, D_4, $\mathbb{Z}_6$, and $\mathbb{Z}_4$.

Let Γ be the embedding of K_5 illustrated in Figure 9. Any transposition which fixes vertex 5 is induced by a reflection through the plane containing the three vertices fixed by the transposition. To see that any transposition involving vertex 5 can be achieved, consider the automorphism (15). Pull $\overline{51}$

through the triangle $\overline{234}$ and then turn over the embedding so that vertex 5 is at the top, vertex 1 is in the centre and vertices 3 and 4 are switched. Now reflect in the plane containing vertices 1, 5, and 2 in order to switch vertices 3 and 4 back. All other transpositions involving vertex 5 can be induced by a similar sequence of moves. Hence $\mathrm{TSG}(\Gamma) \cong \mathrm{S}_5$.

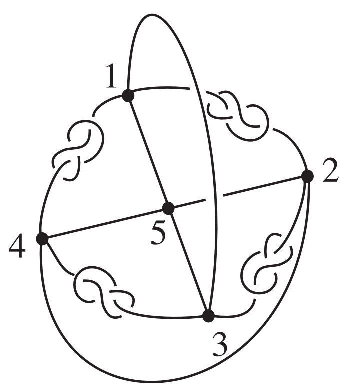
Figure 9. $\mathrm{TSG}(\Gamma) \cong \mathrm{S}_5$.



We create a new embedding Γ' from Figure 9 by adding the achiral figure eight knot, 4_1, to all edges containing vertex 5. Now every homeomorphism of (S^3, Γ') fixes vertex 5, yet all transpositions fixing vertex 5 are still possible. Thus $\mathrm{TSG}(\Gamma') \cong \mathrm{S}_4$.

In order to prove D_4 is realizable for K_5 consider the embedding Γ illustrated in Figure 10. Every homeomorphism of (S^3, Γ) takes $\overline{1234}$ to itself, so $\mathrm{TSG}(\Gamma) \leq \mathrm{D}_4$. The automorphism (1234) is induced by rotating the graph by $90°$ about a vertical line through vertex 5, then reflecting in the plane containing the vertices $1, 2, 3, 4$, and finally isotoping the knots into position. Furthermore, reflecting in the plane containing $\overline{153}$ or $\overline{254}$ and then isotoping the knots into position yields the transposition (24) or (13) respectively. Hence $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_4$.

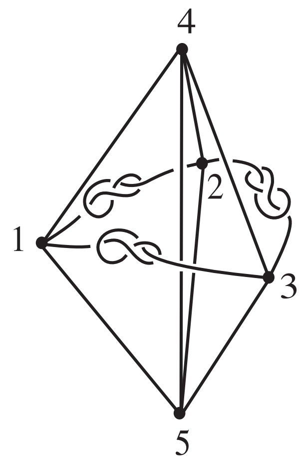
Figure 10. $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_4$.



We obtain a new embedding Γ' by replacing the invertible 4_1 knots in Figure 10 with the knot 12_{427}, which is positive achiral but non-invertible [16]. Since 12_{427} is neither negative achiral nor invertible, no homeomorphism of (S^3, Γ') can invert $\overline{1234}$. Thus $\mathrm{TSG}(\Gamma') \cong \mathbb{Z}_4$.

Next let Γ denote the embedding of K_5 illustrated in Figure 11. Every homeomorphism of (S^3, Γ) takes $\overline{123}$ to itself, so $\mathrm{TSG}(\Gamma) \leq \mathrm{D}_6$. The 3-cycle (123) is induced by a rotation. Each transposition involving only vertices 1, 2, and 3 is induced by a reflection in the plane containing $\overline{45}$ and the remaining fixed vertex followed by an isotopy. The transposition (45) is induced by a reflection in the plane containing vertices 1, 2 and 3 followed by an isotopy. Thus $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_6$, generated by (123), (23), and (45).

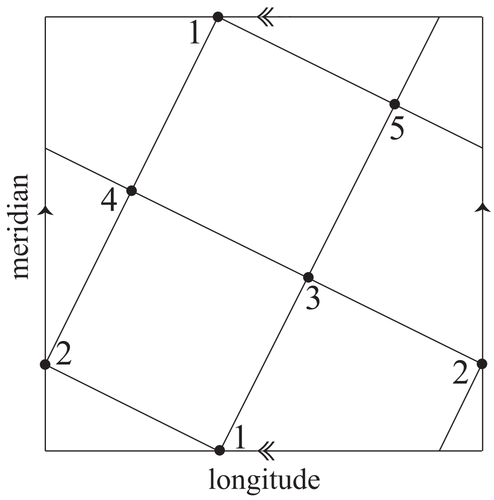
Figure 11. $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_6$.



We obtain a new embedding Γ' by replacing the 4_1 knots in Figure 11 by 12_{427} knots. Then the triangle $\overline{123}$ cannot be inverted. Thus $\mathrm{TSG}(\Gamma') \cong \mathbb{Z}_6$, generated by (123) and (45).

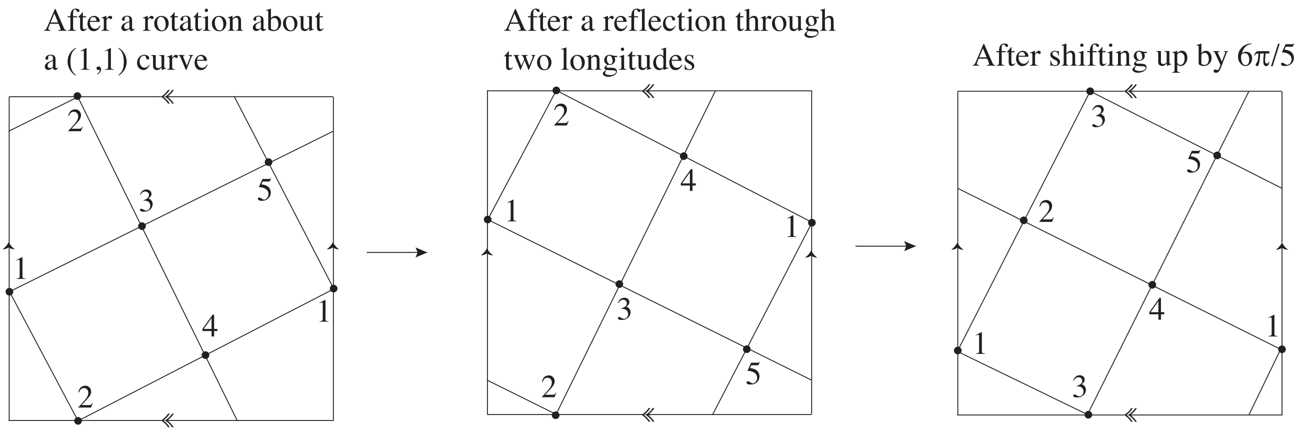
It is more difficult to show that $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ is realizable for K_5, so we define our embedding in two steps. First we create an embedding Γ of K_5 on a torus T that is standardly embedded in S^3. In Figure 12, we illustrate Γ on a flat torus. Let f denote a glide rotation of S^3 which rotates the torus longitudinally by $4\pi/5$ while rotating it meridinally by $8\pi/5$. Thus f takes Γ to itself inducing the automorphism (12345).

Figure 12. The embedding Γ of K_5 in a torus.



Let g denote the homeomorphism obtained by rotating S^3 about a $(1, 1)$ curve on the torus T, followed by a reflection through a sphere meeting T in two longitudes, and then a meridional rotation of T by $6\pi/5$. In Figure 13, we illustrate the step-by-step action of g on T, showing that g takes Γ to itself inducing (2431).

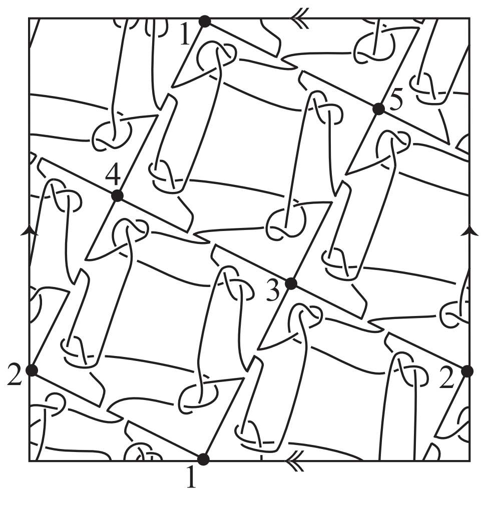
Figure 13. The action of g on Γ.



The homeomorphisms f and g induce the automorphisms $\phi = (12345)$ and $\psi = (2431)$ respectively. Observe that $\phi^5 = \psi^4 = 1$ and $\psi\phi = \phi\psi^2$. Thus $\langle \phi, \psi \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \leq \mathrm{TSG}(\Gamma) \leq \mathrm{S}_5$. Note however that the embedding in Figure 12 is isotopic to the embedding of K_5 in Figure 9. Thus $\mathrm{TSG}(\Gamma) \cong \mathrm{S}_5$.

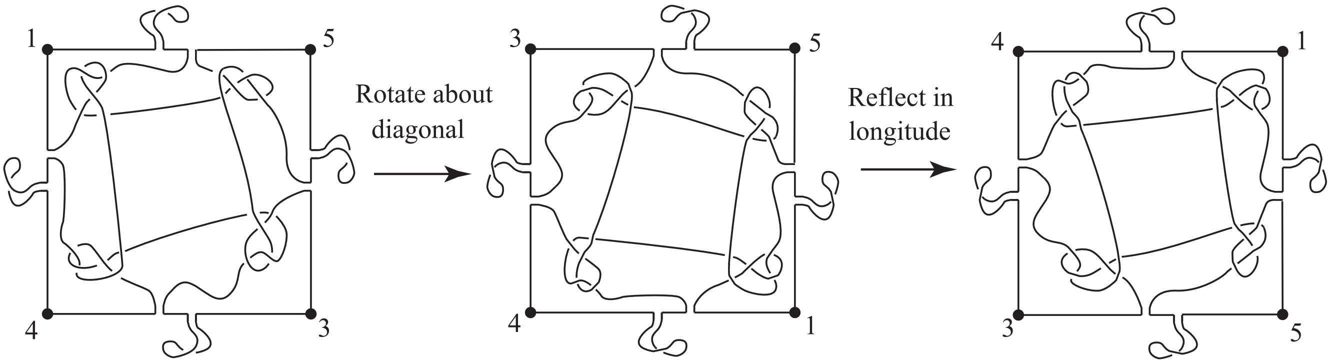
In order to obtain the group $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, we now consider the embedding Γ' of K_5 whose projection on a torus is illustrated in Figure 14. Observe that the projection of Γ' in every square of the grid given by Γ on the torus is identical. Thus the homeomorphism f which took Γ to itself inducing the automorphism $\phi = (12345)$ on Γ also takes Γ' to itself inducing ϕ on Γ'.

Figure 14. Projection of Γ' on the torus.



Recall that g was the homeomorphism of (S^3, Γ) obtained by rotating S^3 about a $(1, 1)$ curve on the torus T, followed by a reflection through a sphere meeting T in two longitudes, and then a meridional rotation of T by $6\pi/5$. In order to see what g does to Γ', we focus on the square $\overline{1534}$ of Γ'. Figure 15 illustrates a rotation of the square $\overline{1534}$ about a diagonal, then a reflection of the square across a longitude. The result of these two actions takes the projection of the knot $\overline{1534}$ to an identical projection. Thus after rotating the torus meridionally by $6\pi/5$, we see that g takes Γ' to itself inducing the automorphism $\psi = (2431)$. It now follows that $\mathbb{Z}_5 \rtimes \mathbb{Z}_4 \leq \mathrm{TSG}(\Gamma') \leq S_5$.

Figure 15. Effect of g on the square $\overline{1534}$.



In order to prove that $\mathrm{TSG}(\Gamma') \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$, we need to show $\mathrm{TSG}(\Gamma') \not\cong \mathrm{S}_5$. We prove this by showing that the automorphism (15) cannot be induced by a homeomorphism of (S^3, Γ').

From Figure 15 we see that the square $\overline{1534}$ is the knot $4_1\#4_1\#4_1\#4_1$. In order to see what would happen to this knot if the transposition (15) were induced by a homeomorphism of (S^3, Γ'), we consider the square $\overline{5134}$. In Figures 16 and 17 we isotop $\overline{5134}$ to a projection with only 10 crossings. This means that $\overline{5134}$ cannot be the knot $4_1\#4_1\#4_1\#4_1$. It follows that the automorphism (15) cannot be induced by a homeomorphism of (S^3, Γ'). Hence $\mathrm{TSG}(\Gamma') \not\cong S_5$. However, the only subgroup of S_5 that contains $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ and is not S_5 is the group $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Thus in fact $\mathrm{TSG}(\Gamma') \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

Figure 16. The knot $\overline{5134}$ projected on the torus.

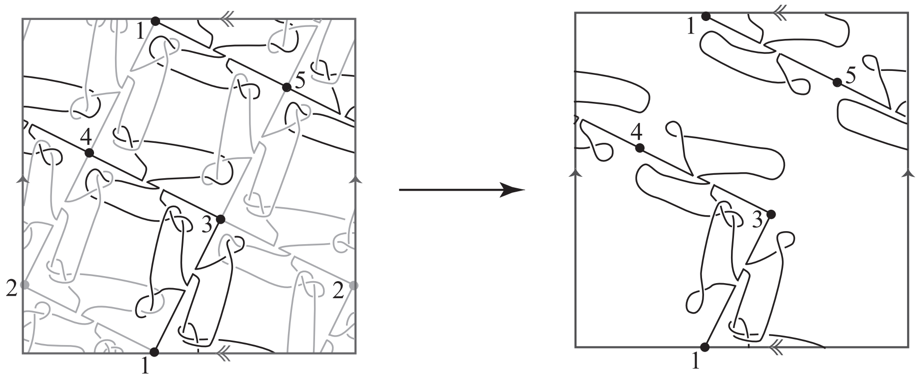
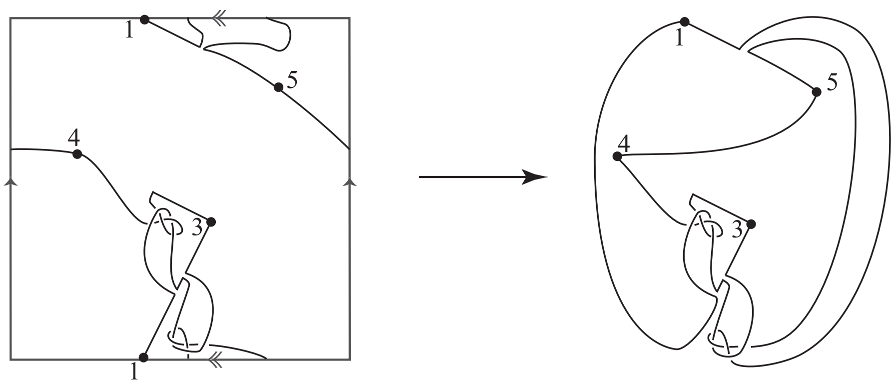


Figure 17. A projection of $\overline{5134}$ on the torus after an isotopy, followed by a projection of $\overline{5134}$ on a plane.



Thus every subgroup of $\mathrm{Aut}(K_5)$ is realizable for K_5. Table 4 summarizes our results for $\mathrm{TSG}(K_5)$.

Table 4. Non-trivial realizable groups for K_5.

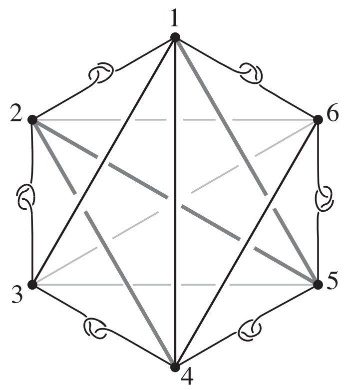
Subgroup	Realizable	Reason
S_5	Yes	By Figure 9
A_5	Yes	Positively realizable
S_4	Yes	By modifying Figure 9
A_4	Yes	Positively realizable
D_6	Yes	By Figure 11
D_5	Yes	Positively realizable
D_4	Yes	By Figure 10
D_3	Yes	Positively realizable
D_2	Yes	Positively realizable
$\mathbb{Z}_6$	Yes	By modifying Figure 11
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	Yes	By Figure 14
$\mathbb{Z}_5$	Yes	Positively realizable
$\mathbb{Z}_4$	Yes	By modifying Figure 10
$\mathbb{Z}_3$	Yes	Positively realizable
$\mathbb{Z}_2$	Yes	Positively realizable

6. Topological Symmetry Groups of K_6

The following is a complete list of all the subgroups of $\mathrm{Aut}(K_6) \cong \mathrm{S}_6$:
S_6, A_6, S_5, A_5, $S_2[S_3]$ ($B[A]$ represents a wreath product of A by B.), $\mathrm{S}_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathrm{D}_3 \times \mathrm{D}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\mathrm{D}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, D_6, D_5, D_4, $\mathrm{D}_4 \times \mathbb{Z}_2$, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [17] and independently verified using the program GAP).

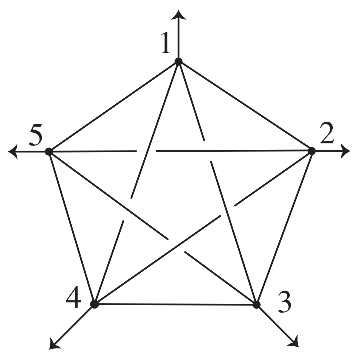
Consider the embedding Γ of K_6 illustrated in Figure 18. There are three paths in Γ on different levels that look like the letter "Z" which are highlighted in Figure 18. The top Z-path is $\overline{3146}$, the middle Z-path is $\overline{4251}$, and the bottom Z-path is $\overline{5362}$. The knotted cycle $\overline{123456}$ must be setwise invariant under every homeomorphism of Γ, and hence $\mathrm{TSG}_+(\Gamma) \leq D_6$. The automorphism (123456) is induced by a glide rotation that cyclically permutes the Z-paths. Consider the homeomorphism obtained by rotating Γ by $180°$ about the line through vertices 2 and 5 and then pulling the edges $\overline{13}$ and $\overline{46}$ to the top level while pushing the lower edges down. The result of this homeomorphism is that the top Z-path $\overline{3146}$ goes to the top Z-path $\overline{1364}$, the middle Z-path $\overline{4251}$ goes to to middle Z-path $\overline{6253}$, and the bottom Z-path $\overline{5362}$ goes to the bottom Z-path $\overline{5142}$. Thus the homeomorphism leaves Γ setwise invariant inducing the automorphism $(13)(46)$. It follows that $\mathrm{TSG}_+(\Gamma) = \langle (123456), (13)(46) \rangle \cong \mathrm{D}_6$. Finally, since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups $\mathbb{Z}_6$, D_3, $\mathbb{Z}_3$, D_2 and $\mathbb{Z}_2$ are positively realizable for K_6.

Figure 18. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_6$.



Consider the embedding, Γ of K_6 illustrated in Figure 19 with vertex 6 at infinity. The automorphisms (13524) and $(25)(34)$ are induced by rotations. Also since $\overline{13524}$ is the only 5-cycle which is knotted, $\overline{13524}$ is setwise invariant under every homeomorphism of (S^3, Γ). Hence $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_5$. Also since $\overline{15}$ is not pointwise fixed under any homeomorphism, by the Subgroup Theorem, $\mathbb{Z}_5$ is positively realizable for K_6.

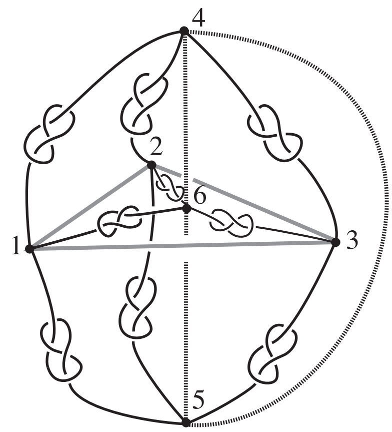
Figure 19. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_5$.



Next consider the embedding, Γ of K_6 illustrated in Figure 20. The automorphisms $(123)(456)$ and $(123)(465)$ are induced by glide rotations and $(45)(12)$ is induced by turning the figure upside down. Also if we consider the circles $\overline{123}$ and $\overline{465}$ as cores of complementary solid tori, then $(14)(25)(36)$ is induced by an orientation preserving homeomorphism that switches the two solid tori.

Observe that every homeomorphism of (S^3, Γ) takes the pair of triangles $\overline{123} \cup \overline{456}$ to itself, since this is the only pair of complementary triangles not containing knots. The automorphism group of the union of two triangles is $S_2[S_3]$ [18]. Thus $\mathrm{TSG}_+(\Gamma) \leq S_2[S_3]$. Note that the transpositions (12) and (45) are each induced by a reflection followed by an isotopy. Thus $\mathrm{TSG}(\Gamma) \cong S_2[S_3]$, since $(123)(456)$, $(123)(465)$, (12) and $(14)(25)(36)$ generate $S_2[S_3]$. However, by the Complete Graph Theorem, $\mathrm{TSG}_+(\Gamma) \ncong S_2[S_3]$. Thus $\mathrm{TSG}_+(\Gamma)$ must be an index 2 subgroup of $S_2[S_3]$ containing $f = (123)(456)$, $g = (123)(465)$, $\phi = (45)(12)$ and $\psi = (14)(25)(36)$. Observe that $\phi\psi$ is the involution $(42)(51)(36)$, and f commutes with ψ and also $f\phi\psi = \phi\psi f^{-1}$, while g commutes with $\phi\psi$ and $g\psi = \psi g^{-1}$. Thus $S_2[S_3] \gneq \mathrm{TSG}_+(\Gamma) \geq \langle f, \phi\psi\rangle \times \langle g, \psi\rangle \cong \mathrm{D}_3 \times \mathrm{D}_3$. It follows that $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3 \times \mathrm{D}_3$.

Figure 20. $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_3 \times \mathrm{D}_3$.



The subgroup $\langle f, g, \psi\rangle$ is isomorphic to $\mathrm{D}_3 \times \mathbb{Z}_3$ because ψ commutes with f and $g\psi = \psi g^{-1}$. We add the non-invertible knot 8_{17} to every edge of the triangles $\overline{123}$ and $\overline{456}$ to obtain an embedding Γ_1. Now the automorphism $\phi = (45)(12)$ cannot be induced by an orientation preserving homeomorphism of (S^3, Γ_1). However, f, g, and ψ are still induced by orientation preserving homeomorphisms. Thus $\mathrm{TSG}_+(\Gamma_1) \cong \mathrm{D}_3 \times \mathbb{Z}_3$ since $\mathrm{D}_3 \times \mathbb{Z}_3$ is a maximal subgroup of $\mathrm{D}_3 \times \mathrm{D}_3$.

Also $\langle f, g, \phi\rangle$ is isomorphic to $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ because $f\phi = \phi f^{-1}$ and $g\phi = \phi g^{-1}$. Again starting with Γ in Figure 20, we place 5_2 knots on the edges of the triangle $\overline{123}$ so that ψ is no longer induced. Thus creating and embedding Γ_2 with $\mathrm{TSG}_+(\Gamma_2) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ since $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ is a maximal subgroup of $\mathrm{D}_3 \times \mathrm{D}_3$.

Finally $\langle f, g\rangle$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. If we place equivalent non-invertible knots on each edge of the triangle $\overline{123}$ and a another set (distinct from the first set) of equivalent non-invertible knots on each edge of $\overline{456}$ we obtain an embedding Γ_3 with $\mathrm{TSG}_+(\Gamma_3) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ since $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a maximal subgroup of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

We summarize our results on positively realizability for K_6 in Table 5. Note in the last few lines of the table we list multiple groups per line, since all of these groups are not positively realizable for the same reason.

By adding appropriate equivalent chiral knots to each edge, every group which is positively realizable for K_6 is also realizable for K_6. Thus we only need to determine realizability for the groups S_6, A_6, S_5, A_5, $\mathrm{S}_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, D_4, $\mathrm{D}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that in Figure 20 we already determined that $S_2[S_3]$ is realizable for K_6.

Let Γ_4 be the embedding of K_6 illustrated in Figure 20 with a left handed trefoil added to each edge of $\overline{123}$ and a right handed trefoil added to each edge of $\overline{456}$. The pair of triangles are setwise invariant since no other edges contain trefoils. Both $(123)(456)$ and $(123)(465)$ are induced by homeomorphisms of (Γ_4, S^3). Also if we reflect in the plane containing vertices 4, 5, 6, and 1 then all the trefoils switch from left-handed to right-handed and vice versa. If we then interchange the complementary solid tori which have the triangles as cores followed by an isotopy, we obtain an orientation reversing homeomorphism that induces the order 4 automorphism $(14)(25)(36)(23) = (14)(2536)$. Now $\langle (14)(2536), (123)(456), (123)(465)\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$.

Table 5. Non-trivial positively realizable groups for K_6.

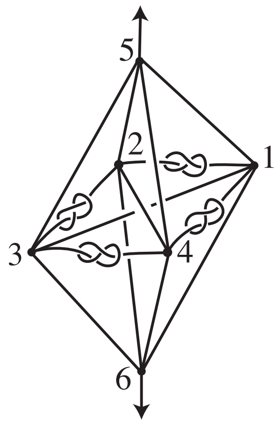
Subgroup	**Positively Realizable**	**Reason**
A_5	No	By A_5 Theorem
S_4	No	By S_4 Theorem
A_4	No	By A_4 Theorem
D_6	Yes	By Figure 18
D_5	Yes	By Figure 19
D_4	No	By Lemma 2
$\mathrm{D}_3 \times \mathrm{D}_3$	Yes	By Figure 20
$\mathrm{D}_3 \times \mathbb{Z}_3$	Yes	By modifying Figure 20
D_3	Yes	By Subgroup Theorem
D_2	Yes	By Subgroup Theorem
$\mathbb{Z}_6$	Yes	By Subgroup Theorem
$\mathbb{Z}_5$	Yes	By Subgroup Theorem
$\mathbb{Z}_4$	No	By Lemma 2
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Yes	By modifying Figure 20
$\mathbb{Z}_3 \times \mathbb{Z}_3$	Yes	By modifying Figure 20
$\mathbb{Z}_3$	Yes	By Subgroup Theorem
$\mathbb{Z}_2$	Yes	By Subgroup Theorem
S_6, A_6, S_5, $S_2[S_3]$, $\mathrm{S}_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$	No	By Complete Graph Theorem
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, $\mathrm{D}_4 \times \mathbb{Z}_2$	No	By Complete Graph Theorem
$\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	No	By Complete Graph Theorem

We see as follows that $\mathrm{TSG}(\Gamma_4)$ cannot be larger than $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$. Suppose that the automorphism (12) is induced by a homeomorphism f. By Lemma 1, f must be orientation reversing. But $f(\overline{456}) = \overline{456}$, which is impossible because $\overline{456}$ contains only right handed trefoils. Thus $\mathrm{TSG}(\Gamma_4) \ncong S_2[S_3]$. Note that the only proper subgroup of $S_2[S_3]$ containing $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ is $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$. Thus $\mathrm{TSG}(\Gamma_4) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$.

Now let Γ be the embedding of K_6 illustrated in Figure 21. Observe that the linking number $\mathrm{lk}(\overline{135}, \overline{246}) = \pm 1$, but $\mathrm{lk}(\overline{136}, \overline{245}) = 0$. Thus the automorphism (56) cannot be induced by a homeomorphism of (S^3, Γ). Since every homeomorphism of (S^3, Γ) takes $\overline{1234}$ to itself, it follows that $\mathrm{TSG}(\Gamma) \leq \mathrm{D}_4$. The automorphism $(1234)(56)$ is induced by a rotation followed by a reflection and an isotopy. In addition the automorphism $(14)(23)(56)$ is induced by turning the figure upside down. Thus $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_4$ generated by the automorphisms $(1234)(56)$ and $(14)(23)(56)$.

Now let Γ' be obtained from Figure 21 by replacing the knot 4_1 with the non-invertible and positively achiral knot 12_{427}. Then the square $\overline{1234}$ can no longer be inverted. In this case $(1234)(56)$ generates $\mathrm{TSG}(\Gamma')$ and thus $\mathrm{TSG}(\Gamma') \cong \mathbb{Z}_4$.

Figure 21. $\mathrm{TSG}(\Gamma) \cong \mathrm{D}_4$.



For the next few groups we will use the following lemma.

4-Cycle Theorem. *[19] For any embedding Γ of K_6 in S^3, and any labelling of the vertices of K_6 by the numbers 1 through 6, there is no homeomorphism of (S^3, Γ) which induces the automorphism (1234).*

Consider the subgroup $\mathbb{Z}_5 \rtimes \mathbb{Z}_4 \leq \mathrm{Aut}(K_6)$. The presentation of $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ as a subgroup of S_6 gives the relation $x^{-1}yx = y^2$ for some elements $x, y \in \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ of orders 4 and 5 respectively. Suppose that for some embedding Γ of K_6, we have $\mathrm{TSG}(\Gamma) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Without loss of generality, we can assume that $y = (12345)$ satisfies the relation $x^{-1}yx = y^2$. By the 4-Cycle Theorem, any order 4 element of $\mathrm{TSG}(\Gamma)$ must be of the form $x = (abcd)(ef)$. However, there is no element in $\mathrm{Aut}(K_6)$ of the form $x = (abcd)(ef)$ that together with $y = (12345)$ satisfies this relation. Thus there can be no embedding Γ of K_6 in S^3 such that $\mathrm{TSG}(\Gamma) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

Now consider the subgroup $\mathbb{Z}_4 \times \mathbb{Z}_2 \leq \mathrm{Aut}(K_6)$. By the 4-Cycle Theorem, without loss of generality we may assume that if $\mathrm{TSG}(\Gamma)$ contains an element of order 4 for some embedding Γ of K_6, then $\mathrm{TSG}(\Gamma)$ contains the element $(1234)(56)$. Computation shows that the only transposition in $\mathrm{Aut}(K_6)$ that commutes with $(1234)(56)$ is (56), which cannot be an element of $\mathrm{TSG}(\Gamma)$ since this would imply that (1234) is an element of $\mathrm{TSG}(\Gamma)$. Furthermore the only order 2 element of $\mathrm{Aut}(K_6)$ that commutes with $(1234)(56)$ and is not a transposition is $(13)(24)$, which is already in the group generated by $(1234)(56)$. Thus there is no embedding Γ of K_6 in S^3 such that $\mathrm{TSG}(\Gamma)$ contains the group $\mathbb{Z}_4 \times \mathbb{Z}_2$. This rules out all of the groups $\mathrm{S}_4 \times \mathbb{Z}_2$, $\mathrm{D}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ as possible topological symmetry groups for embeddings of K_6 in S^3.

For the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ we will use the following result.

Conway Gordon. *[20] For any embedding Γ of K_6 in S^3, the mod 2 sum of the linking numbers of all pairs of complementary triangles in Γ is 1.*

Now suppose that for some embedding Γ of K_6 in S^3 we have $\mathrm{TSG}(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It can be shown that the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \leq \mathrm{Aut}(K_6)$ contains three disjoint transpositions. Without loss of generality we can assume that $\mathrm{TSG}(\Gamma)$ contains (13), (24), and (56), which are induced by homeomorphisms h, f, and g of (S^3, Γ) respectively. Since any three vertices of Γ determine a pair of disjoint triangles, we can use a triple of vertices to represent a pair of disjoint triangles. For example, we use the triple 123 to denote the pair of triangles $\overline{123}$ and $\overline{456}$. With this notation, the orbits of the ten pairs of disjoint triangles in K_6 under the group $\langle(13), (24), (56)\rangle$ are:

$$\{123, 143\}, \{124, 324\}, \{125, 325, 145, 126\}, \{135, 136\}$$

Since h, f, and g are homeomorphisms of (S^3, Γ) the links in a given orbit all have the same (mod 2) linking number. Since each of these orbits has an even number of pair of triangles, this contradicts Conway Gordon. Thus $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathrm{TSG}(\Gamma)$. Hence $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is not realizable for K_6

Table 6 summarizes our realizability results for K_6. Recall that for $n = 4$ and $n = 5$ every subgroup of S_n is realizable for K_n. However, as we see from Table 6, this is not true for $n = 6$.

Table 6. Non-trivial realizable groups for K_6.

Subgroup	**Realizable**	**Reason**
S_6	No	$\mathrm{TSG}_+(K_6)$ cannot be S_6 or A_6
A_6	No	$\mathrm{TSG}_+(K_6)$ cannot be A_6
S_5	No	$\mathrm{TSG}_+(K_6)$ cannot be S_5 or A_5
A_5	No	$\mathrm{TSG}_+(K_6)$ cannot be A_5
$\mathrm{S}_4 \times \mathbb{Z}_2$	No	$\mathrm{TSG}_+(K_6)$ cannot be $\mathrm{S}_4 \times \mathbb{Z}_2$ or S_4
S_4	No	$\mathrm{TSG}_+(K_6)$ cannot be S_4 or A_4
$A_4 \times \mathbb{Z}_2$	No	$\mathrm{TSG}_+(K_6)$ cannot be $A_4 \times \mathbb{Z}_2$ or A_4
A_4	No	$\mathrm{TSG}_+(K_6)$ cannot be A_4
D_6	Yes	Positively realizable
D_5	Yes	Positively realizable
$\mathrm{D}_4 \times \mathbb{Z}_2$	No	$\mathrm{TSG}_+(K_6)$ cannot be $\mathrm{D}_4 \times \mathbb{Z}_2$, D_4, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
D_4	Yes	By Figure 21
$S_2[S_3]$	Yes	By Figure 20
$\mathrm{D}_3 \times \mathrm{D}_3$	Yes	Positively realizable
$\mathrm{D}_3 \times \mathbb{Z}_3$	Yes	Positively realizable
D_3	Yes	Positively realizable
D_2	Yes	Positively realizable
$\mathbb{Z}_6$	Yes	Positively realizable
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	No	By 4-Cycle Theorem
$\mathbb{Z}_5$	Yes	Positively realizable
$\mathbb{Z}_4 \times \mathbb{Z}_2$	No	By 4-Cycle Theorem
$\mathbb{Z}_4$	Yes	By modifying Figure 21
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	Yes	By modifying Figure 20
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Yes	Positively realizable
$\mathbb{Z}_3 \times \mathbb{Z}_3$	Yes	Positively realizable
$\mathbb{Z}_3$	Yes	Positively realizable
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	No	By Conway Gordon Theorem
$\mathbb{Z}_2$	Yes	Positively realizable

7. Conclusions

We have classified all groups which can occur as the topological symmetry group or orientation preserving topological symmetry group of an embedded complete graph with no more than six vertices. Our results show that a number of groups can occur as a topological symmetry group but not as an orientation preserving topological symmetry group for a particular K_n. This gives us a collection of groups which can only occur for achiral embeddings of the graph in question.

The topological symmetry group includes all of the symmetries induced by the point group together with any symmetries that occur as the result of any flexibility or rotation of subparts of a structure around specific bonds. Thus the topological symmetry group gives us more information about the symmetries and possible achirality of supramolecular structures than could be obtained from the point group. Since complete graphs with no more than six vertices may occur as supramolecular clusters, these results could be of interest in the future study of supramolecular chirality.

Acknowledgements

The first author would like to thank Claremont Graduate University for its support while he pursued the study of Topological Symmetry Groups for his Ph.D Thesis. The second author would like to thank the Institute for Mathematics and its Applications at the University of Minnesota for its hospitality while she was a long term visitor in the fall of 2013.

Author contribution

The authors worked on all sections of this article together.

Conflicts of Interest

The authors declare no conflict of interest.

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