CLASSIFICATION OF TOPOLOGICAL SYMMETRY GROUPS OF K_n

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Abstract. In this paper we complete the classification of topological symmetry groups for complete graphs K_n by characterizing which K_n can have a cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ where m is odd, as its topological symmetry group.

1. Introduction

Characterizing the symmetries of a molecule is an important step in predicting its chemical behaviour. Chemists have long used the group of rigid symmetries, known as the *point group*, as a means of representing the symmetries of a molecule. However, molecules which are flexible or partially flexible may have symmetries which are not included in the point group. Jon Simon [10] introduced the concept of the *topological symmetry group* in order to study symmetries of such non-rigid molecules. The topological symmetry group provides a way to classify, not only the symmetries of molecular graphs, but the symmetries of any graph embedded in S^3.

We define the topological symmetry group as follows. Let γ be an abstract graph, and let $\mathrm{Aut}(\gamma)$ denote the automorphism group of γ. Let Γ be the image of an embedding of γ in S^3. The *topological symmetry group* of Γ, denoted by $\mathrm{TSG}(\Gamma)$, is the subgroup of $\mathrm{Aut}(\gamma)$ which is induced by homeomorphisms of the pair (S^3, Γ). The *orientation preserving topological symmetry group* of Γ, denoted by $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{Aut}(\gamma)$ which is induced by orientation preserving homeomorphisms of the pair (S^3, Γ). In this paper we are only concerned with $\mathrm{TSG}_+(\Gamma)$, and thus for simplicity we abuse notation and refer to the group $\mathrm{TSG}_+(\Gamma)$ simply as the *topological symmetry group* of Γ.

Frucht [8] showed that every finite group is the automorphism group of some connected graph. Since every graph can be embedded in S^3, it is natural to ask whether every finite group can be realized as $\mathrm{TSG}_+(\Gamma)$ for some connected graph Γ embedded in S^3. Flapan, Naimi, Pommersheim, and Tamvakis proved in [6] that the answer to this question is "no", and proved

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that there are strong restrictions on which groups can occur as topological symmetry groups. For example, it was shown that $\mathrm{TSG}_+(\Gamma)$ can never be the alternating group A_n for $n > 5$.

The special case of topological symmetry groups of complete graphs is interesting to consider because a complete graph K_n has the largest automorphism group of any graph with n vertices. In [7], Flapan, Naimi, and Tamvakis characterized which finite groups can occur as topological symmetry groups of embeddings of complete graphs in S^3, as follows. Note that D_m denotes the dihedral group with $2m$ elements.

Complete Graph Theorem. [7] *A finite group H is $\mathrm{TSG}_+(\Gamma)$ for some embedding of Γ of a complete graph in S^3 if and only if H is isomorphic to a finite cyclic group, a dihedral group, A_4, S_4, A_5, or a finite subgroup of $D_m \times D_m$ for some odd m.*

Observe that the Complete Graph Theorem does not tell us which K_n can have a given group H as their topological symmetry group. In previous work, the first three authors have characterized which complete graphs can have A_4, S_4 or A_5 as their topological symmetry groups [1, 2, 5]. In Theorems 1, 2 and 3 of this paper we complete the classification of topological symmetry groups for complete graphs K_n by characterizing which K_n can have a cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ (for m odd) as its topological symmetry group. For a complete list of these subgroups, see the $D_m \times D_m$ Lemma in Section 2.

Theorem 1. *Let $G = \mathbb{Z}_m$ or D_m. A complete graph K_n, $n > 6$, has an embedding Γ such that $\mathrm{TSG}_+(\Gamma) = G$ if and only if one of the following holds:*

(1) *$m \geq 4$ is even, and $n \equiv 0 \pmod{m}$.*
(2) *$m \geq 3$ is odd and $n \equiv 0, 1, 2, 3 \pmod{m}$.*
(3) *$m = 2$, $G = D_m$, and $n \equiv 0, 1, 2 \pmod{4}$.*
(4) *$m = 2$ and $G = \mathbb{Z}_m$.*

Theorem 2. *Let $G = \mathbb{Z}_r \times \mathbb{Z}_s$ or $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ where r, s are odd and $\gcd(r, s) > 1$. A complete graph K_n, with $n > 6$, has an embedding Γ with $\mathrm{TSG}_+(\Gamma) = G$ if and only if one of the following conditions holds:*

(1) *$rs|n$.*
(2) *$\gcd(r, s) = 3$ and $rs|(n - 3)$.*
(3) *$G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $9|(n - 6)$.*
(4) *$G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and $18|(n - 6)$.*

Theorem 3. *Let $G = \mathbb{Z}_r \times D_s$ or $D_r \times D_s$ where $r, s \geq 3$ are odd. A complete graph K_n, with $n > 6$, has an embedding Γ with $\mathrm{TSG}_+(\Gamma) = G$ if and only if one of the following conditions holds:*

(1) *$2rs|n$.*
(2) *$G = \mathbb{Z}_3 \times D_3$ and $18|(n - 6)$.*
(3) *$G = D_3 \times D_3$ and $36|(n - 6)$.*

The paper is organized as follows. In Section 2, we provide previous results which we will use in our proofs. In Section 3 we will prove Theorem 1. In Section 4, we will prove the necessity of the conditions for Theorems 2 and 3, and in Section 5 we will complete the proofs of these theorems. Finally, in Section 6 we will combine our results here with our earlier results in [5] to give a complete list of all possible topological symmetry groups for complete graphs on 20 or fewer vertices.

2. Background

Our first three results provide tools to prove that certain subgroups of $\mathrm{Aut}(K_n)$ *cannot* be realized as topological symmetry groups. The Automorphism Theorem [3] establishes restrictions on which automorphisms of a complete graph can be realized by homeomorphisms, and hence could possibly be elements of the topological symmetry group of an embedding. The Isometry Theorem [7] allows us to assume that, roughly speaking, the topological symmetry group of an embedded complete graph is induced by isometries. This allows us to use properties of isometries to prove that certain topological symmetry groups are impossible. Finally, the No-D_2 Lemma [1] allows us to rule out topological symmetry groups containing D_2 for the graphs K_{4r+3}.

Automorphism Theorem. [3] *Let K_n be a complete graph on $n > 6$ vertices and let ϕ be an automorphism of K_n. Then there is an embedding Γ of K_n in S^3 such that ϕ is induced by an orientation preserving homeomorphism h of (S^3, Γ) of order m if and only if the cycles and fixed vertices of ϕ can be described by one of the following:*

(1) *$m > 2$ is even, all cycles of ϕ are of order m, and ϕ fixes no vertices.*
(2) *$m = 2$, all cycles of ϕ are of order m, and ϕ fixes at most two vertices.*
(3) *m is odd, all cycles of ϕ are of order m, and ϕ fixes at most three vertices.*
(4) *m is an odd multiple of 3 and $m > 3$, all cycles of ϕ are of order m except one of order 3, and ϕ fixes no vertices.*

Isometry Theorem. [7] *Let Ω be an embedding of some K_m in S^3 such that* $\mathrm{TSG}_+(\Omega)$ *is not a cyclic group of odd order. Then K_m can be re-embedded in S^3 as Γ such that* $\mathrm{TSG}_+(\Omega) \leq \mathrm{TSG}_+(\Gamma)$ *and* $\mathrm{TSG}_+(\Gamma)$ *is induced by an isomorphic finite subgroup of* $\mathrm{SO}(4)$.

Remark: It follows from the proof of Theorem 1 that the Isometry Theorem also holds if $\mathrm{TSG}_+(\Omega)$ is a cyclic group of odd order.

No-D_2 Lemma. [1] *There is no embedding Γ of K_{4r+3} in S^3 such that $D_2 \leq \mathrm{TSG}_+(\Gamma)$.*

The next few results provide tools for proving that certain groups *can* be realized as topological symmetry groups. The Realizability Lemma [7] allows us to realize certain subgroups of $\mathrm{Aut}(K_n)$ that do not fix any vertices as topological symmetry groups. The Edge Embedding Lemma [5] allows us to construct embeddings whose topological symmetry groups contain specified subgroups by just embedding the vertices, subject to certain restrictions.

Realizability Lemma. [7] *Let γ be a 3-connected graph, and let $H \leq \mathrm{Aut}(\gamma)$ be such that no vertex is fixed by any non-trivial element of H. Suppose that H is isomorphic to a subgroup $G \leq \mathrm{SO}(4)$ such that every involution of G has non-empty fixed point set and if $g \in G$ is an involution then no $h \in G$ distinct from g has $\mathrm{fix}(h) = \mathrm{fix}(g)$. Then there is an embedding Γ of γ in S^3 such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by G.*

Edge Embedding Lemma. [5] *Let G be a finite subgroup of $\mathrm{Diff}_+(S^3)$, and let γ be a graph whose vertices are embedded in S^3 as a set V which is invariant under G such that G induces a faithful action on γ. Suppose that adjacent pairs of vertices in V satisfy the following hypotheses:*

1. *If a pair is pointwise fixed by non-trivial elements $h, g \in G$, then $\mathrm{fix}(h) = \mathrm{fix}(g)$.*
2. *For each pair $\{v, w\}$ in the fixed point set C of some non-trivial element of G, there is an arc $A_{vw} \subseteq C$ bounded by $\{v, w\}$ whose interior is disjoint from V and from any other such arc $A_{v'w'}$*
3. *If a point in the interior of some A_{vw} or a pair $\{v, w\}$ bounding some A_{vw} is setwise invariant under an $f \in G$, then $f(A_{vw}) = A_{vw}$.*
4. *If a pair is interchanged by some $g \in G$, then the subgraph of γ whose vertices are pointwise fixed by g can be embedded in a proper subset of a circle.*
5. *If a pair is interchanged by some $g \in G$, then $\mathrm{fix}(g)$ is non-empty, and $\mathrm{fix}(h) \neq \mathrm{fix}(g)$ if $h \neq g$.*

Then the embedding of the vertices of γ can be extended to the edges of γ in S^3 such that the resulting embedding of γ is setwise invariant under G.

To complement these results, the Subgroup Lemma [4] gives us conditions under which a subgroup of the topological symmetry group of an embedding of K_n can itself be realized as the topological symmetry group for some other embedding of K_n.

Subgroup Lemma. [4] *Suppose Γ is an embedding of a graph in S^3 such that either (i) Γ is 3-connected and contains an edge e which is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$; or (ii) Γ is isomorphic to K_n for some $n > 6$ and $\mathrm{TSG}_+(\Gamma)$ is a finite cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ for some odd m. Then for every (possibly trivial) subgroup H of $\mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of K_n such that $H = \mathrm{TSG}_+(\Gamma')$.*

Finally, the following lemma lists the subgroups of $D_m \times D_m$ for m odd.

$D_m \times D_m$ Lemma. [4] *Let $m \geq 3$ be odd, and let $G \leq D_m \times D_m$. Then G is isomorphic to one of the following groups where $r, s \geq 3$ are odd: $\mathbb{Z}_2$, $\mathbb{Z}_r$,*

$\mathbb{Z}_{2r}$, D_2, D_r, D_{2r}, $\mathbb{Z}_r \times \mathbb{Z}_s$, $D_r \times \mathbb{Z}_s$, $D_r \times D_s$, or $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ (where, if $x \in \mathbb{Z}_r \times \mathbb{Z}_s$ and $y \in \mathbb{Z}_2$, $xy = yx^{-1}$).

3. The cyclic and dihedral groups

We begin with the proof of Theorem 1, which describes when a complete graph has an embedding whose topological symmetry group is either cyclic or dihedral. Throughout, the notation $\langle S \rangle_G$ denotes the orbit of a set of points S under the action of a group G.

Proof. Since $n > 6$, and $\mathbb{Z}_m$ and D_m each contain an element of order m, the necessity of the conditions for $m \geq 3$ is immediate from the Automorphism Theorem. If $m = 2$, the restriction that $n \not\equiv 3 \pmod 4$ for D_2 comes from the No-D_2 Lemma.

For the sufficiency, we will first show that for the values of m and n given in the hypotheses of the theorem, $D_m \leq \mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_n. Then we will conclude that for $G = \mathbb{Z}_m$ or D_m, $\mathrm{TSG}_+(\Gamma') = G$ for some embedding Γ' of K_n. Note that part (4) of the theorem follows from parts (1) and (2): Given any $n > 6$, we let $m = n$, so $n \equiv 0 \pmod m$, and hence $D_m \leq \mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_n. As $\mathbb{Z}_2 \leq D_m$, it follows from the Subgroup Lemma that $\mathrm{TSG}_+(\Gamma') = \mathbb{Z}_2$ for some embedding Γ' of K_n. Thus we will not discuss part (4) any further.

First, consider part (1) of the theorem; i.e., $m \geq 4$ is even, and $n = mr$ for some r. We describe a subgroup of $\mathrm{SO}(4)$ isomorphic to D_m. Let h_1 be a rotation of S^3 of order m with fixed point set a geodesic circle C_1, and h_2 be a rotation of S^3 of order $m/2$ with fixed point set a geodesic circle C_2 that is setwise fixed by h_1. Let h be the glide rotation $h_1 \circ h_2$ of order m, and let a be an involution of S^3 whose fixed point set A intersects each of C_1 and C_2 orthogonally in two points. Then the group $H = \langle h, a \rangle$ generated by h and a is isomorphic to D_m.

To embed the vertices of K_n, choose $\lfloor r/2 \rfloor$ points $v_1, \ldots, v_{\lfloor r/2 \rfloor}$ in $S^3 - (C_1 \cup C_2 \cup \langle A \rangle_H)$ which have disjoint orbits under H, and embed $2m\lfloor r/2 \rfloor$ vertices as the orbits of the v_i's under H. If r is even, we've embedded all the vertices; if r is odd, then we have m vertices that have not yet been embedded. Embed one of these vertices as some $v \in A - (C_1 \cup C_2)$, and embed the remaining vertices as the points in $\langle v \rangle_H$. Note that $h^{m/2}$ is an involution that fixes A setwise but not pointwise; thus, there will be two vertices embedded on each of the $m/2$ circles in $\langle A \rangle_H$.

We have embedded the vertices of K_n as a set V which is invariant under H so that H induces a faithful action on V. In the case where r is even, we embedded all vertices in $S^3 - (C_1 \cup C_2 \cup \langle A \rangle_H)$, so no vertex is fixed by a nontrivial element of H. Hence, by the Realizability Lemma, we obtain an embedding Γ of K_n such that $\mathrm{TSG}_+(\Gamma) = D_m$, as desired. In the case where r is odd, we use the Edge Embedding Lemma to embed the edges of K_n; so we need to verify that the five conditions of the lemma are satisfied. We will use the following claim to verify one of the conditions.

Claim: For all $g \in H$, $g(A)$ either equals A or is disjoint from A.

Proof of Claim: Let $g \in H$ be such that $g(A)$ intersects A. We wish to show $g(A) = A$. Let $x \in A \cap g(A)$. Then x is fixed by both a and gag^{-1}. Now, $agag^{-1} = h^i$ for some positive $i \leq m$ (since, in any dihedral group, the product of two reflections is not a reflection). So x is fixed by h^i. Now, $h^i = h_1^i h_2^i$ has fixed points only if $i = m/2$ or $i = m$; and, in either case, $h^i(A) = A$. Since $agag^{-1} = h^i$ and $a^{-1} = a$, we get $ag^{-1} = g^{-1}ah^i$; so $ag^{-1}(A) = g^{-1}ah^i(A) = g^{-1}(A)$, i.e., $g^{-1}(A)$ is fixed by a. Therefore $g^{-1}(A) \subset A$, which implies $g^{-1}(A) = A$ since A is a circle and g is an isometry. Therefore $g(A) = A$, completing the proof of the claim.

Let $w = h^{m/2}(v)$; then v and w are the two vertices on the circle A. Now, any pair of vertices that are pointwise fixed by a nontrivial element of H are of the form $\{g(v), g(w)\}$ for some $g \in H$; and, for each such pair, gag^{-1} is the unique nontrivial element of H that fixes the pair pointwise. So condition (1) of the Edge Embedding Lemma is satisfied. Let A_{vw} be one of the two arcs on A from v to w. For each pair of vertices $\{v', w'\} = \{g(v), g(w)\}$, let $A_{v'w'} = g(A_{vw})$. Any two such arcs are disjoint since, by the above claim, for all $g \in H$, $g(A)$ either equals A or is disjoint from A. Hence condition (2) is satisfied as well. The only nontrivial elements of H that fix a point in the interior of A_{vw} or setwise fix the pair of vertices $\{v, w\}$ are $h^{m/2}$ and a, both of which setwise fix A_{vw}. And similarly for other pairs $\{v', w'\}$; hence condition (3) is satisfied. The pair $\{v, w\}$ is interchanged by only one element of H, namely $h^{m/2}$; and this element pointwise fixes the circle C_1 and no vertices. So, by a similar argument for the other pairs $\{v', w'\}$, we see that conditions (4) and (5) are also satisfied. Hence, by the Edge Embedding Lemma, there is an embedding Γ of K_n such that $D_m \leq \mathrm{TSG}_+(\Gamma)$.

Next, consider part (2) of the theorem, i.e., $m \geq 3$ is odd, and $n = mr + k$ for some r and for some $k \in \{0, 1, 2, 3\}$ (where $k \neq 3$ if $m = 3$). This time, we let h be a rotation of S^3 of order m with fixed point set a circle C, and let a be an involution of S^3 whose fixed point set A intersects C orthogonally in two points. Then $H = \langle h, a \rangle = D_m$. We embed k vertices on C as follows: if $k = 1$, one vertex is embedded as a point of $C \cap A$; if $k = 2$, one vertex is embedded in $C - A$, and the other is embedded as the image of the first under a; if $k = 3$, we embed two vertices as for $k = 2$, and the third as for $k = 1$. For the remaining $n - k = mr$ vertices, we proceed as follows. Embed $\lfloor r/2 \rfloor$ points in the complement of $C \cup A$ such that they have distinct orbits, giving a total of $2m\lfloor r/2 \rfloor$ points under the action of H. If r is even, this allows us to embed all of the mr remaining vertices, as desired. If r is odd, we can embed only $mr - m$ vertices this way. The remaining m vertices are embedded as the orbit of a point in $A - C$. Note that in this case, since m is odd, for each circle $A' \subset \langle A \rangle_H$, only one vertex is on $A' - C$.

Thus, for part (2) of the theorem, we have embedded the vertices of K_n as a set V which is invariant under H so that H induces a faithful action on V. We now use the Edge Embedding Lemma to embed the edges of K_n.

The fixed point sets of the elements of H consist of C and the circles in $\langle A\rangle_H$. Any two of these circles meet at the two points $C\cap A$. Since we have placed a vertex at at most one of these points, two elements of H can fix a pair of vertices only if they share the same fixed point set. So condition (1) of the Edge Embedding Lemma is satisfied.

Now, if v and w are distinct vertices pointwise fixed by an element of H, then either $\{v,w\}\subset C$ or $\{v,w\}\subset A'$ for some $A'\in\langle A\rangle_H$. Since $k\leq 3$, the vertices on C can all be connected by arcs in $C-V$ whose interiors are disjoint. The vertex in $C\cap A$ (if there is one) can be connected to each of the vertices (if any) in $\langle A-C\rangle_H$ by arcs in $\langle A-C\rangle_H$ whose interiors are all disjoint from each other and from the arcs on C. So condition (2) is satisfied. And any element of H which fixes a point in the interior of one of these arcs, or interchanges its endpoints, also fixes the arc setwise. So condition (3) is satisfied. The only elements of H which interchange vertices are involutions. Each involution pointwise fixes at most two vertices of K_n and the edge between them, so condition (4) is satisfied. Finally, the fixed point set of each involution is non-empty, and is not the fixed point set of any other element of H, so condition (5) is satisfied. Hence, by the Edge Embedding Lemma, there is an embedding Γ of the edges and vertices of K_n such that $D_m\leq \mathrm{TSG}_+(\Gamma)$.

Finally, consider part (3) of the theorem, i.e., $m=2$ and $n=4r+k$ for some r, and some $k\in\{0,1,2\}$. We use the same construction as for part (2), with $H=\langle h,a\rangle$, C, and A as before. The vertices are embedded as in part (2), i.e., if $k=1$, embed one vertex as a point in $C\cap A$; if $k=2$, embed two vertices as a point in $C-A$ and its image under a; and embed the remaining $4r$ vertices as the orbit under H of r points in the complement of $C\cup A$. The conditions of the Edge Embedding Lemma are verified as in the construction for part (2), except that they are simpler here since $\langle A\rangle_H=\{A\}$ and there are no vertices on $A-C$. Hence, again there is an embedding Γ of K_n such that $D_m\leq \mathrm{TSG}_+(\Gamma)$.

We have shown that for all values of m and n in the hypothesis of the theorem, there is an embedding Γ of K_n such that $H=D_m\leq \mathrm{TSG}_+(\Gamma)$. In all of our constructions of Γ, either (i) there is a vertex which is not fixed by any non-trivial element of H, or (ii) there is a vertex which is fixed by an involution in H but not by every element of H: case (i) occurs in parts (1) and (2) if $n=mr+k$ and $r\geq 2$, and also in part (3) since $n=4r+k\geq 7$ and $k\leq 2$, which implies $r\geq 2$; case (ii) occurs in parts (1) and (2) if $n=mr+k$ and $r=1$. Let $G=\mathbb{Z}_m$ or D_m. In case (i), by the Subgroup Lemma, we can re-embed K_n as Γ' such that $\mathrm{TSG}_+(\Gamma')=G$. In case (ii), by [5, Lemma 2.2], $\mathrm{TSG}_+(\Gamma)$ does not contain A_4 and hence is not a polyhedral group. So, by the Complete Graph Theorem and the Subgroup Lemma, again there is a re-embedding Γ' of K_n such that $\mathrm{TSG}_+(\Gamma')=G$. $\square$

4. Some necessary conditions for subgroups of $D_m \times D_m$

We now look at the subgroups of $D_m \times D_m$ (for m odd) that involve products of cyclic and dihedral groups. In this section, we will prove the necessity of some of the conditions in Theorems 2 and 3. We begin with the definition of a **standard presentation** of certain subgroups of $D_m \times D_m$:

- $\langle \rho, \sigma \rangle$ is a *standard presentation* of $\mathbb{Z}_r \times \mathbb{Z}_s$ if $\mathbb{Z}_r \times \mathbb{Z}_s = \langle \rho, \sigma \rangle$, where $\text{order}(\rho) = r$, $\text{order}(\sigma) = s$, and ρ commutes with σ.
- $\langle \rho, \sigma, \phi \rangle$ is a *standard presentation* of $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ if $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2 = \langle \rho, \sigma, \phi \rangle$, where $\text{order}(\rho) = r$, $\text{order}(\sigma) = s$, $\text{order}(\phi) = 2$, $\rho\sigma = \sigma\rho$, $\phi\rho = \rho^{-1}\phi$, and $\phi\sigma = \sigma^{-1}\phi$.
- $\langle \rho, \sigma, \beta \rangle$ is a *standard presentation* of $\mathbb{Z}_r \times D_s$ if $\mathbb{Z}_r \times D_s = \langle \rho, \sigma, \beta \rangle$, where $\text{order}(\rho) = r$, $\text{order}(\sigma) = s$, $\text{order}(\beta) = 2$, ρ commutes with β and σ, and $\beta\sigma = \sigma^{-1}\beta$.
- $\langle \rho, \alpha, \sigma, \beta \rangle$ is a *standard presentation* of $D_r \times D_s$ if $D_r \times D_s = \langle \rho, \alpha, \sigma, \beta \rangle$, where $\text{order}(\rho) = r$, $\text{order}(\sigma) = s$, $\text{order}(\alpha) = \text{order}(\beta) = 2$, each of ρ and α commutes with σ and β, $\alpha\rho = \rho^{-1}\alpha$, and $\beta\sigma = \sigma^{-1}\beta$.

We first prove some necessary conditions for realizing $\mathbb{Z}_r \times D_s$ and $D_r \times D_s$. These conditions will depend on some elementary facts about isometries of S^3.

Lemma 1. *Let τ and β be bijective maps from a set to itself such that $\beta\tau = \tau^{\pm 1}\beta$. Then $\beta(\text{fix}(\tau)) = \text{fix}(\tau)$.*

Proof. If $z \in \text{fix}(\tau)$, then $\beta(z) = \beta\tau(z) = \tau^{\pm 1}\beta(z)$. Therefore, $\beta(z) \in \text{fix}(\tau)$, so $\beta(\text{fix}(\tau)) \subseteq \text{fix}(\tau)$. By a similar argument, $\beta^{-1}(\text{fix}(\tau)) \subseteq \text{fix}(\tau)$. Since β is a bijection, this means $\text{fix}(\tau) \subseteq \beta(\text{fix}(\tau))$, and therefore $\beta(\text{fix}(\tau)) = \text{fix}(\tau)$. □

Lemma 2. *Let $G = \langle \tau, \beta \rangle$ be a finite subgroup of* $\text{SO}(4)$ *which leaves a simple closed curve $C \subset S^3$ setwise invariant. Suppose* $\text{order}(\tau) > 2$. *Then τ and β commute if and only if β preserves the orientation of C.*

Proof. Since G leaves C setwise invariant, there is a neighborhood of C which G leaves setwise invariant and the action of G on this neighborhood is conjugate to the action on the normal bundle of C. Now since the normal bundle of a simple closed curve has a canonical trivialization by parallel transport, there is an invariant product neighborhood $C \times D^2$ whose product structure is preserved by G. Thus we can write the restrictions $\tau|_{C \times D^2}$ and $\beta|_{C \times D^2}$ as product maps (τ_1, τ_2) and (β_1, β_2) respectively, where τ_1 and β_1 are maps of C and τ_2 and β_2 are maps of D^2.

We make the following observations about the elements of any finite group action of a circle or disk:

1. If an element is orientation reversing, then it has order 2.
2. Any pair of orientation preserving elements commute.

(3) Suppose that g is an element with order greater than 2. Then any element which commutes with g is orientation preserving.

Since $\text{order}(\tau) > 2$, for some $k = 1$ or 2, $\text{order}(\tau_k) > 2$. Hence by Observation 1, τ_k must be orientation preserving. Since τ is orientation preserving, it follows that both τ_1 and τ_2 must be orientation preserving.

Suppose that β preserves the orientation of C, then so does β_1. Since β is orientation preserving, it follows that β_2 is also orientation preserving. By Observation 2, τ_j and β_j commute for each j. Now $\tau\beta\tau^{-1}\beta^{-1}$ has finite order and pointwise fixes $C \times D^2$, so it must be trivial. Hence τ and β commute.

Conversely, suppose that τ and β commute. Then for each j, τ_j and β_j commute. Recall that k was chosen so that $\text{order}(\tau_k) > 2$. Since τ_k and β_k commute, by Observation 3, β_k is orientation preserving. Again since β is orientation preserving, this means that both β_1 and β_2 are as well. In particular, β preserves the orientation of C. □

Lemma 3. *Any nontrivial, orientation preserving, odd order rotation of S^3 setwise fixes exactly two geodesic circles.*

Proof. Let s be an orientation preserving, odd order rotation of of S^3; then $s = s'|_{S^3}$ for some odd order element $s' \in \text{SO}(4)$. Hence s' is conjugate to a rotation r that fixes the x_1x_2-plane (P) setwise, and the x_3x_4-plane (P') pointwise. Since r and s' are conjugate, it suffices to show that r does not setwise fix any geodesic circle other than the two in P and P'. Every geodesic circle of S^3 is the intersection of S^3 with a plane through the origin in $\mathbb{R}^4$. Suppose r setwise fixes a geodesic $P'' \cap S^3$ distinct from $P \cap S^3$ and $P' \cap S^3$. As r is a linear transformation of $\mathbb{R}^4$, it must also setwise fix the plane P''. Since P and P' are orthogonal complements of each other and P'' is distinct from P', P'' does not lie in the orthogonal complement of P; hence P'' contains a vector v whose projection onto P is nonzero. So we can write $v = w + w'$, where w and w' are the projections of v onto P and P' respectively, and w is nonzero. Now, $r(v) \in r(P'') = P''$, and P'' is a subspace, so $r(v) - v \in P''$. Furthermore, $r(w') = w'$ since r pointwise fixes P'. So $u = r(v) - v = r(w) - w$ is a nonzero vector in P since $w \neq 0$ and r rotates w by a nonzero angle. Now, as u lies in both P and P'', $r(u)$ also lies in both P and P''. Hence $r(u)$ is parallel to u; so r must rotate u by a multiple of π, which contradicts the fact that s has odd order. So r does not setwise fix any geodesic circle other than $P \cap S^3$ and $P' \cap S^3$. □

We use these lemmas to prove the following result about subgroups of SO(4) isomorphic to $\mathbb{Z}_r \times D_s$.

Lemma 4. *Let $\langle \rho, \sigma, \beta \rangle$ be a standard presentation of $\mathbb{Z}_r \times D_s \leq \text{SO}(4)$, where $r, s \geq 3$ and r is odd. Then for every positive $i < \text{order}(\rho)$ and positive $j < \text{order}(\sigma)$, ρ^i and σ^j are fixed point free.*

Proof. Suppose toward contradiction that $\mathrm{fix}(\sigma^j)$ is nonempty for some positive $j < s$. Then $\mathrm{fix}(\sigma^j)$ is a circle C. Since $\beta\sigma = \sigma^{-1}\beta$, by Lemma 1, $\beta(C) = C$. Furthermore, since $s \geq 3$, by Lemma 2, β must reverse the orientation of C. So β fixes exactly two points in C and is a rotation by π about a circle B that intersects C in those two points.

Since ρ commutes with both β and σ^j, by Lemma 1 it setwise fixes each of B and C. Hence it setwise fixes $B \cap C$, which consists of two points. As ρ has odd order, it cannot exchange these two points, so it must fix each of them. Now, for each circle B and C, since ρ setwise fixes the circle and fixes two points on it and has odd order, it must pointwise fix the circle, so ρ fixes $B \cup C$. This implies ρ is trivial, which is a contradiction.

Now, suppose toward contradiction that $\mathrm{fix}(\rho^j)$ is nonempty for some positive $j < r$. Then $\mathrm{fix}(\rho^j)$ is a circle. Since β and σ each commute with ρ^j, by Facts 1 and 2 they each setwise fix and preserve the orientation of $\mathrm{fix}(\rho^j)$. But, by Lemma 2, this contradicts the hypothesis that $\beta\sigma = \sigma^{-1}\beta$. □

We now apply these results to prove restrictions on the complete graphs which can be embedded so that their topological symmetry group contains $\mathbb{Z}_r \times D_s$ or $D_r \times D_s$.

Lemma 5. *Let $n > 6$ and let $r, s \geq 3$ be odd. Suppose Γ is an embedding of K_n such that $\mathbb{Z}_r \times D_s$ is induced on Γ by an isomorphic subgroup $G \leq \mathrm{SO}(4)$.*

1. *If some non-trivial element of G fixes a vertex of Γ, then $r = s = 3$ and $2rs|(n-6)$. Otherwise, $2rs|n$.*
2. *If $\langle \rho, \sigma, \beta \rangle$ is a standard presentation of G, then there exist geodesic circles A and B in S^3 such that: (i) $G(A \cup B) = A \cup B$, (ii) β interchanges A with B, and (iii) if a non-trivial element of G fixes a vertex of Γ, then each of A and B contains exactly three vertices of Γ.*
3. *If $H = \langle \rho, \alpha, \sigma, \beta \rangle \leq \mathrm{SO}(4)$ is a standard presentation of $D_r \times D_s$ and induces an isomorphic subgroup on Γ, then, in addition to the conclusions of part (2) above, we have $H(A \cup B) = A \cup B$ and α interchanges A with B.*

Proof. By [7, Lemma 2], G satisfies the Involution Condition (defined in [7]). Therefore, by [7, Proposition 3], G preserves a standard Hopf fibration of S^3, since otherwise it would be isomorphic to one of the polyhedral groups. And by [7, Lemma 3], there exist two fibers A and B such that $\{A, B\}$ is setwise invariant under G. Let $\langle \rho, \sigma, \beta \rangle$ be a standard presentation of G. Since ρ and σ each have odd order, they cannot interchange A with B. Therefore they must setwise fix each of A and B.

Suppose β fixes A setwise. Then, since β commutes with ρ, by Lemma 2, β must preserve the orientation of A. On the other hand, $\beta\sigma = \sigma^{-1}\beta$, and $s \geq 3$; so by Lemma 2, β must reverse the orientation of A. Thus we get a contradiction. Since $\beta(A \cup B) = A \cup B$ and $\beta(A) \neq A$, β must interchange A with B.

We claim no involution in G can fix any vertices. Suppose toward contradiction that an involution γ fixes a vertex v_0. Then $\text{fix}(\gamma)$ is a circle X. By hypothesis, ρ commutes with every element of G, and in particular with γ. Hence, by Lemma 1, $\rho(X) = X$, which gives $\langle v_0 \rangle_\rho \subset X$. By Lemma 4, for every positive $i < r$, ρ^i has no fixed points, so $|\langle v_0 \rangle_\rho| = r \geq 3$. But, by the Automorphism Theorem, γ cannot fix more than two vertices, which is a contradiction.

If no vertex of Γ is fixed by any nontrivial element of G, then every vertex orbit under G has size $|G| = 2rs$, and $2rs|n$. Suppose a vertex v_0 is fixed by some nontrivial element $g_0 \in G$ that is not an involution. Since $\text{fix}(g_0)$ is nonempty, it is a circle C. By replacing g_0 by its square if necessary, we can assume $g_0 = \rho^i \sigma^j$ for some i, j. So g_0 commutes with both ρ and σ. Hence, by Lemma 1, $\langle \rho, \sigma \rangle$ setwise fixes C. By Lemma 4, for every positive $k < r$ and $l < s$, ρ^k and σ^l have no fixed points; so $\langle v_0 \rangle_{\langle \rho, \sigma \rangle} \subset C$ contains at least three vertices. On the other hand, as g_0 is nontrivial, by the Automorphism Theorem its fixed point set C cannot contain more than 3 vertices. It follows that $\langle v_0 \rangle_{\langle \rho, \sigma \rangle}$ consists of exactly 3 vertices. Thus, ρ^3 and σ^3 must each fix v_0. Therefore, by Lemma 4, $r = s = 3$.

Now, $g_0 = \rho^i \sigma^j$ is a rotation of odd order about C. So, by Lemma 3, it setwise fixes at most two geodesic circles. As g_0 setwise fixes each of A and B, we must have $C = A$ or $C = B$. Thus v_0 is in A or B. Recall that the orbit of v_0 under $\langle \rho, \sigma \rangle$ contains exactly three vertices; and β interchanges A with B. It follows that $|\langle v_0 \rangle_G| = 6$. Furthermore, these 6 vertices are the only vertices that are fixed by any nontrivial element of G since any other such vertex would also have an orbit of size 6 with 3 on A and 3 on B, contradicting that C contains only 3 vertices. Thus every vertex that's not in $\langle v_0 \rangle_G$ is fixed only by the trivial element and hence has an orbit of size $|G| = 2rs$. Therefore $2rs|(n-6)$, which finishes part (1) of the lemma. The above also establishes part (2) of the lemma.

To prove part (3), suppose $\langle \rho, \alpha, \sigma, \beta \rangle$ is a standard presentation of $D_r \times D_s$ and is induced by an isomorphic subgroup $H \leq \text{SO}(4)$ on Γ. But applying the argument given above for $\langle \rho, \sigma, \beta \rangle$ to $\langle \rho, \alpha, \sigma \rangle$ instead, we see that α also interchanges A with B, and hence $H(A \cup B) = A \cup B$. □

Finally, we prove restrictions on when the topological symmetry group can be $D_3 \times D_3$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Lemma 6. *Assume that K_n, with $n > 6$, has an embedding Γ such that $D_3 \times D_3 \leq \text{TSG}_+(\Gamma)$. If $18|(n-6)$, then $36|(n-6)$.*

Proof. Suppose toward contradiction that there is an embedding Γ of K_n such that $\text{TSG}_+(\Gamma) = D_3 \times D_3$, and $n - 6$ is an odd multiple of 18. Since $n > 6$, K_n is 3-connected; so $\text{TSG}_+(\Gamma)$ is induced by an isomorphic subgroup H of $\text{SO}(4)$. Let $H = \langle \rho, \alpha, \sigma, \beta \rangle$ be a standard presentation of $D_3 \times D_3$.

By Lemma 5, there exist geodesic circles A and B such that $H(A \cup B) = A \cup B$ and each of these two circles contains exactly three vertices of Γ. Thus these six vertices are setwise invariant under H. Let Γ' be the embedding of $K_{n'}$, where $n' = n-6$, obtained by removing these six vertices from Γ. Then Γ' is invariant under H, and hence under its subgroup $\mathbb{Z}_r \times D_s = \langle \rho, \sigma, \beta \rangle$. Since $18|n'$, by Lemma 5, no element of $\langle \rho, \sigma, \beta \rangle$ fixes any vertex of Γ'. Similarly, no element of $\langle \rho, \alpha, \sigma \rangle$ fixes any vertex of Γ'. It follows that only elements of $H - (\langle \rho, \sigma, \beta \rangle \cup \langle \rho, \alpha, \sigma \rangle)$ can possibly fix any vertex of Γ'.

Now, some vertex in Γ' must be fixed by some nontrivial element in H, since otherwise the orbit size of every vertex would be a multiple of $|H| = 36$, contradicting that n' is an odd multiple of 18. Let $\phi = \alpha\beta$. Then every element of $H - (\langle \rho, \sigma, \beta \rangle \cup \langle \rho, \alpha, \sigma \rangle)$ is of the form $\rho^i \sigma^j \alpha\beta$ for some i and j. Note that $(\rho^{2i}\sigma^{2j})\alpha\beta(\rho^{2i}\sigma^{2j})^{-1} = \rho^{4i}\sigma^{4j}\alpha\beta = \rho^i\sigma^j\alpha\beta$. So every element of $H - (\langle \rho, \sigma, \beta \rangle \cup \langle \rho, \alpha, \sigma \rangle)$ is an involution conjugate to $\phi = \alpha\beta$. Thus, some element conjugate to ϕ fixes a vertex of Γ'. Therefore ϕ itself fixes some vertex $v \in \Gamma'$.

By Lemma 5, α and β each interchange A with B. It follows that ϕ setwise fixes each of A and B. Since each of A and B contains three of the vertices of $\Gamma - \Gamma'$ and $\mathrm{order}(\phi) = 2$, ϕ must fix one vertex on each of A and B. These two vertices are distinct from v since the latter is in Γ' while the former are in $\Gamma - \Gamma'$. Therefore ϕ fixes three vertices of Γ, which is a contradiction since by the Automorphism Theorem no involution has three fixed points. □

Lemma 7. *Assume that $n > 6$ and K_n has an embedding Γ such that $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \leq \mathrm{TSG}_+(\Gamma)$. If $9|(n-6)$, then $18|(n-6)$.*

Proof. Assume $9|(n-6)$, but n is odd. By the Isometry Theorem, we may assume that $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ is generated by a finite group of orientation-preserving isometries of (S^3, Γ). Let $\langle \rho, \sigma, \phi \rangle$ be a standard presentation of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Observe that, since n is odd, ϕ must fix an odd number of vertices; by the Automorphism Theorem, this means ϕ fixes exactly one vertex. Also, it follows from [1] that ρ and σ each fix either 0 or 3 vertices, $\rho(\mathrm{fix}(\sigma)) = \mathrm{fix}(\sigma)$, $\sigma(\mathrm{fix}(\rho)) = \mathrm{fix}(\rho)$, and the fixed point sets are disjoint. It also follows from [1] that there are at most two disjoint sets of three vertices which are setwise invariant under both ρ and σ (including the fixed vertices). Let X be the union of all such sets; then $|X| = 0, 3$ or 6, and includes the fixed vertices of ρ and σ. We prove below, as in [1], that $|X| = 6$. Let Γ be the result of removing the vertices of X (and all adjacent edges) from the graph Γ. Then, by the Automorphism Theorem, ρ partitions the vertices of Γ into 3-cycles, as does σ. But σ does not fix any of the cycles of ρ (since we've removed the vertices of X), so it permutes the 3-cycles of ρ (since ρ and σ commute, σ takes each cycle of ρ to some cycle of ρ). Hence the number of 3-cycles must be a multiple of 3, so 9 divides the number of vertices of Γ', which is $n - |X|$. Since $9|(n-6)$, this means $|X| = 6$, so there are exactly two disjoint sets, $V = \{v_0, v_1, v_2\}$ and $W = \{w_0, w_1, w_2\}$, which are setwise invariant under both ρ and σ.

Now there are three possibilities (up to renaming ρ, σ, V and W):

(1) ρ fixes V pointwise and permutes W, and σ fixes W pointwise and permutes V, or
(2) ρ fixes V pointwise and permutes W, and σ permutes both V and W, or
(3) both ρ and σ permute both V and W.

Case 1: Assume ρ fixes V pointwise and permutes W, and σ fixes W pointwise and permutes V. Then $\rho\phi(v_i) = \phi\rho^{-1}(v_i) = \phi(v_i)$, so $\phi(v_i)$ is a fixed vertex of ρ, and hence $\phi(v_i) \in V$. So ϕ fixes V setwise. Since ϕ has order 2 and $|V| = 3$, ϕ must fix an odd number of vertices in V. So, by the Automorphism Theorem, ϕ fixes exactly one vertex of V. Similarly, using σ, ϕ fixes exactly one vertex of W. But then ϕ fixes two vertices, which contradicts our earlier observation that it must fix exactly one vertex.

Case 2: Assume ρ fixes V pointwise and permutes W, and σ permutes both V and W. As in Case 1, ϕ must fix V setwise, and fixes one vertex of V. Let $W' = \phi(W)$, with $w_i' = \phi(w_i)$. Then $\rho(w_i') = \rho\phi(w_i) = \phi\rho^{-1}(w_i) = \phi(w_j) = w_j'$, so ρ fixes W' setwise. Similarly, σ fixes W' setwise. But V and W are the only sets of three vertices which are setwise invariant under both ρ and σ. $W' \neq V$, since $\phi(W') = W$ and $\phi(V) = V$. So we must have $W' = W$, and hence ϕ fixes W setwise. But then we get a contradiction as in Case 1.

Case 3: Assume both ρ and σ permute both V and W. As in Case 2, $\phi(V)$ is either V or W, and $\phi(W)$ is either V or W. So either $\phi(V) = V$ and $\phi(W) = W$, or $\phi(V) = W$ and $\phi(W) = V$. The first case leads to a contradiction as in Case 1, so we may assume that $\phi(V) = W$ and $\phi(W) = V$. Without loss of generality, say $\phi(v_0) = w_0$.

Without loss of generality (replacing σ by σ^{-1} if needed), we may assume that ρ and σ have the same action on V. But then $\rho \neq \sigma$ on W (or $\rho\sigma^{-1}$ would fix 6 vertices, and hence be trivial by Smith Theory), and neither ρ nor σ fix W pointwise, so $\rho = \sigma^{-1}$ on W since ρ and σ both have order 3. So, without loss of generality, $\rho(v_i) = \sigma(v_i) = v_{i+1}$, $\rho(w_i) = w_{i+1}$ and $\sigma(w_i) = w_{i-1}$, where the indices are calculated modulo 3.

Then $\phi(v_1) = \phi\rho(v_0) = \rho^{-1}\phi(v_0) = \rho^{-1}(w_0) = w_2$. But, also, $\phi(v_1) = \phi\sigma(v_0) = \sigma^{-1}\phi(v_0) = \sigma^{-1}(w_0) = w_1$. This is a contradiction.

So every case leads to a contradiction, and we conclude that n must be even, so $18|(n-6)$. □

5. Proofs of Theorems 2 and 3

We now prove Theorem 2.

Proof. Since $\gcd(r,s) > 1$, we can rewrite $\mathbb{Z}_r \times \mathbb{Z}_s$ as $\mathbb{Z}_{\gcd(r,s)} \times \mathbb{Z}_{\mathrm{lcm}(r,s)}$, where the first factor is not trivial. Also, since r and s are both odd, $\gcd(r,s)$ and

$\mathrm{lcm}(r, s)$ are also odd. Therefore, by redefining r and s if necessary, we may assume $r|s$.

The result for $\mathbb{Z}_r \times \mathbb{Z}_s$ then follows from [1] (including part (3)). Since $\mathbb{Z}_r \times \mathbb{Z}_s$ is a subgroup of $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$, the necessity of most of the conditions for $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ follows; the last condition is required by Lemma 7.

For the sufficiency, we consider the subgroup of SO(4) isomorphic to $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ generated as follows. Let α be a rotation of order r about a geodesic circle A, β be a rotation of order s about a geodesic circle B disjoint from A which is setwise fixed by α, and ϕ be a rotation of order 2 about a geodesic circle C which intersects each of A and B in two points. Then $G = \langle \alpha, \beta, \phi \rangle \cong (\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$.

Case 1: We first consider the case when $rs|n$ (i.e, part (1) of the theorem). If $n = (2k)rs$ for some positive integer k, we pick k points in S^3 which are not fixed by any non-trivial elements of G, and whose orbits under G are disjoint. Then each orbit has $2rs$ elements, and we embed the vertices of K_n as the resulting $2rsk$ points. Since none of the vertices is fixed by any nontrivial element of G, the hypotheses of the Realizability Lemma are satisfied.

Now consider the case when $n = (2k + 1)rs$. Embed $(2k)rs$ vertices as described above. For the remaining vertices, pick a point v on $C - (A \cup B)$, so $\phi(v) = v$; then the orbit of v under G contains rs points. Embed the remaining vertices as these points. Since n is odd, by the Automorphism Theorem, any involution fixes only one vertex. So v is the only vertex in $\langle v \rangle_G$ which lies on C. Therefore, at most one vertex of $\langle v \rangle_G$ lies on the fixed point set of any nontrivial element of G, so the first four conditions of the Edge Embedding Lemma are trivially satisfied. Since G has no even-order elements other than involutions, pairs of vertices are only interchanged by involutions, which each have distinct fixed point sets homeomorphic to S^1, so condition (5) is satisfied.

So, if $rs|n$, we have an embedding Γ of K_n such that $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2 \leq \mathrm{TSG}_+(\Gamma)$.

Case 2: Our second case is when $rs \nmid n$, i.e., part (2) of the theorem. We first consider when $n = krs + 3$ and $\gcd(r, s) = 3$. Since we assume that $r|s$, this means $r = 3$. We embed krs vertices as in the previous paragraphs (for k even or odd). Now let w be one point of $B \cap C$, so w is fixed by both β and ϕ (if k is odd, then we choose w to be the point of $B \cap C$ which is in the same component of $C - A$ as the vertex v in the last paragraph). Then the orbit of w under G contains 3 points, all of which are in B; we embed our last three vertices as these three points. Each involution of G fixes one of these three points and interchanges the other two. At most one vertex lies in the intersection of any two fixed point sets of elements of G, so condition (1) of the Edge Embedding Lemma is satisfied. Each pair of vertices fixed by an involution are joined by a unique arc in the fixed point set of the involution whose interior is disjoint from the other vertices and fixed point sets; and the three vertices on B are joined by disjoint arcs in B, so condition (2) is satisfied. The three arcs on B are fixed by β and permuted cyclically by α. Each involution of G fixes one of the vertices on B and interchanges the other two, so it interchanges two of the three arcs, and fixes the third arc setwise (while reversing its endpoints). Therefore, the arcs on B satisfy condition (3); the arcs in the fixed point sets of the involutions are disjoint from the other vertices and fixed points sets, so they also satisfy condition (3). Only involutions interchange pairs of vertices, and each involution fixes at most two vertices of the graph pointwise, so condition (4) is satisfied. Finally, the fixed point sets of the involutions are all circles, and are distinct from the fixed point sets of all other elements of the group, so condition (5) is satisfied.

Finally, we consider when $n = 2krs + 6$ and $r = s = 3$, which is part (4) of the theorem. We embed the vertices as above, except now there are three vertices on each of A and B, and there is no v as above since $2k$ is even. Then, the conditions of the Edge Embedding Lemma are satisfied as above.

So, if $rs \not| n$, we again have an embedding Γ of K_n such that $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2 \leq \mathrm{TSG}_+(\Gamma)$.

So in each case we have an embedding Γ of K_n such that $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2 \leq \mathrm{TSG}_+(\Gamma)$. Since $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$ is not a subgroup of SO(3), the Complete Graph Theorem tells us that $\mathrm{TSG}_+(\Gamma)$ is a subgroup of $D_m \times D_m$ for some odd m. Then, by the Subgroup Lemma, there is an embedding Γ' of K_n such that $\mathrm{TSG}_+(\Gamma') = (\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$. □

Finally, we prove Theorem 3.

Proof. The necessity of most of the conditions is given by Lemma 5; the last condition is required by Lemma 6. It remains to prove that these conditions are sufficient.

Let A and B be the intersections of S^3 with the x_1x_2- and the x_3x_4-planes, respectively, in $\mathbb{R}^4$. Let ρ be a glide rotation obtained by composing a rotation of $2\pi/r$ about A with a rotation of $2\pi/r$ about B, σ a glide rotation obtained by composing a rotation of $2\pi/s$ about A with a rotation of $-2\pi/s$ about B, T the geodesic torus that separates A and B, α a rotation by π about a $(1,-1)$ curve on T, and β a rotation by π about a $(1,1)$ curve on T. Then $\mathrm{order}(\rho) = r$, $\mathrm{order}(\sigma) = s$, $\mathrm{order}(\alpha) = \mathrm{order}(\beta) = 2$, α and β each interchange A with B, each of ρ and α commutes with each of σ and β, α anticommutes with ρ, and β anti-commutes with σ. Thus $G = \langle \rho, \alpha, \sigma, \beta \rangle$ is isomorphic to $D_r \times D_s$.

Case 1: If $2rs|n$, then $n = k(4rs)$ or $n = k(4rs) + 2rs$ for some $k \geq 0$. In the former case, we pick k points $x_1, \cdots, x_k$ disjoint from the fixed point set of all nontrivial elements of G such that any two such points have disjoint orbits under G. We embed the vertices of K_n as the points in the orbits of $x_1, \cdots, x_k$. Since no vertex is fixed by any nontrivial element of G, by the

Realizability Lemma, we get an embedding Γ of K_n such that $\mathrm{TSG}_+(\Gamma) = D_r \times D_s$. We will refer to this set of $k(4rs)$ embedded vertices as X.

Now suppose $n = k(4rs) + 2rs$. Let $\phi = \alpha\beta$. Then $\phi(A) = A$, $\phi(B) = B$, and ϕ is a rotation by π about a geodesic circle C that intersects each of A and B in two points. Thus $C - (A \cup B)$ consists of four arcs, C_1, C_2, C_3, C_4. From our construction we can see that the circle $\mathrm{fix}(\alpha)$ intersects exactly two of these arcs, say C_1 and C_3. In fact, α setwise fixes each of C_1 and C_3 while reversing their orientations, and it interchanges C_2 with C_4.

Let $v \in C_1$ be a point that is not on the fixed point set of any nontrivial element of G other than ϕ. Then v is fixed only by ϕ and the identity; hence $|\langle v\rangle_G| = |G|/2 = 2rs$. We embed $2rs$ vertices of K_n as the orbit of v, and the remaining $k(4rs)$ vertices as X.

Observe that every element of G of even order is an involution conjugate to α, β, or ϕ. Since no vertex is embedded in $\mathrm{fix}(\alpha)$ or $\mathrm{fix}(\beta)$, no element conjugate to α or β fixes any vertices. Since no vertex is embedded in A or B, no element of odd order fixes any vertices. Hence only elements conjugate to ϕ can fix vertices.

We see as follows that ϕ fixes exactly two vertices. Every vertex fixed by ϕ must by in $\langle v\rangle_G$ since the $k(4rs)$ vertices not in the orbit of v are not fixed by any nontrivial elements. Now, $\beta|_C = (\alpha\phi)|_C = \alpha|_C$; hence $\alpha(v) = \beta(v)$. Since the only nontrivial elements of G that take v to a point in C are ϕ, α, and β, we see that $\langle v\rangle_G \cap C$ consists of exactly two points, v and $w = \alpha(v)$. It follows that each element conjugate to ϕ also fixes exactly two vertices.

We now verify that the conditions of the Edge Embedding Lemma are satisfied. Condition (1) is satisfied since only elements conjugate to ϕ fix any vertices and no two such elements have the same fixed point set. Recall that $\alpha(C_1) = C_1$, and we chose v to be in C_1. Hence w is also in C_1. To satisfy Condition (2), let A_{vw} be the arc in C_1 from v to w. Each involution $g\phi g^{-1}$ conjugate to ϕ fixes exactly two vertices, $v' = g(v)$ and $w' = g(w)$; we let $A_{v'w'} = g(A_{vw})$. Then the interior of any such arc $A_{v'w'}$ is disjoint from the set of all embedded vertices as well as from any other such arc. Condition (3) is satisfied since the only elements that fix a point in the interior of $A_{v'w'}$ or setwise fix $\{v', w'\}$ are $g\phi g^{-1}$, $g\alpha g^{-1}$, and $g\beta g^{-1}$, which all setwise fix $A_{v'w'}$. The pair $\{v', w'\}$ is interchanged only by $g\alpha g^{-1}$ and $g\beta g^{-1}$, which do not fix any vertices; this implies Condition (4) is satisfied. Finally, since $g\alpha g^{-1}$ and $g\beta g^{-1}$ have nonempty, distinct fixed point sets, namely $g(\mathrm{fix}(\alpha))$ and $g(\mathrm{fix}(\beta))$, Condition (5) is also satisfied.

So, if $2rs|n$, we have an embedding Γ of K_n such that $D_r \times D_s \leq \mathrm{TSG}_+(\Gamma)$.

Case 2: If $2rs \not| n$, then by Lemma 5 we have that $r = s = 3$ and $18|n-6$, i.e., $n = 6 + 18k$. We will first consider when $k = 2p$ is even, so $n = 6 + 36p$. Let u be a point on A. Then $\langle u\rangle_G$ consists of six points, three on A, three on B. We embed six vertices of K_n as $\langle u\rangle_G$, and the remaining $36p$ vertices as X.

We again verify that the conditions of the Edge Embedding Lemma are satisfied. We need to do this only for pairs of fixed or interchanged vertices in $\langle u\rangle_G$ since we already verified the conditions for pairs in X and there are no pairs of fixed on interchanged vertices one of which is in $\langle u\rangle_G$ and the other in X. Let H denote the set of all elements of the form $\rho^i\sigma^j$, where $i, j \in \{1, 2\}$. There are exactly four such elements; they are rotations by $\pm 2\pi/3$ about either A or B, and hence each fix exactly three vertices. Let u', u'' be a pair of vertices in the orbit of u. If they are pointwise fixed by distinct nontrivial elements $h_1, h_2 \in G$, then we must have u' and u'' both in A or both in B, $h_1^2 = h_2 \in H$, and $\text{fix}(h_1) = \text{fix}(h_2)$. Thus Condition (1) is satisfied. To satisfy Condition (2), we let $A_{u'u''}$ be the arc in $A - \langle u\rangle_G$ or $B - \langle u\rangle_G$ whose boundary is $\{u', u''\}$. Then the interior of any such arc is disjoint from the set of all embedded vertices as well as from any other such arc. Condition (3) is satisfied since only elements in H and elements conjugate to ϕ fix a point in the interior of $A_{u'u''}$ or setwise fix a pair $\{u', u''\}$ bounding $A_{u'u''}$, and all such elements take $A_{u'u''}$ to itself. A pair $\{u', u''\} \subset \langle u\rangle_G$ is interchanged only by elements conjugate to α, β, or ϕ, and any such element fixes at most one vertex. Hence Condition (4) is satisfied. Also, any such element has nonempty fixed point set, and any two such elements have distinct fixed point sets. Hence Condition (5) is satisfied.

So, if $r = s = 3$ and $36|(n-6)$, then there exists an embedding Γ of K_n such that $D_3 \times D_3 \leq \text{TSG}_+(\Gamma)$.

Case 3: It only remains to deal with the case when $r = s = 3$ and $n - 6$ is an odd multiple of 18, i.e., $n = 18k + 6$, where k is odd. Let $G' = \langle\rho, \sigma, \beta\rangle = \mathbb{Z}_r \times D_s \leq G$. Observe that $\langle u\rangle_{G'} = \langle u\rangle_G$. We embed six vertices of K_n as $\langle u\rangle_{G'}$, and the remaining $18k$ vertices as the orbit under G' of k points $x_1, \cdots, x_k$ disjoint from the fixed point sets of all nontrivial elements of G'. Then the conditions of the Edge Embedding Lemma are satisfied by a similar argument as above (but simpler since $\phi \notin G'$). So there exists an embedding Γ of K_n such that $\mathbb{Z}_3 \times D_3 \leq \text{TSG}_+(\Gamma)$.

So in each case we have an embedding Γ of K_n such that $\mathbb{Z}_r \times D_s \leq \text{TSG}_+(\Gamma)$. Since $\mathbb{Z}_r \times D_s$ is not a subgroup of SO(3), the Complete Graph Theorem tells us that $\text{TSG}_+(\Gamma)$ is a subgroup of $D_m \times D_m$ for some odd m. Then, by the Subgroup Lemma, there is an embedding Γ' of K_n such that $\text{TSG}_+(\Gamma') = \mathbb{Z}_r \times D_s$ (and, except in the case when $r = s = 3$ and n is an odd multiple of 18, an embedding Γ'' of K_n such that $\text{TSG}_+(\Gamma') = D_r \times D_s$). □

6. Examples

In this section we will apply Theorems 1, 2 and 3 and the results below to help give examples of the complete list of possible topological symmetry groups for various complete graphs. Note that our results enable one to compute such examples completely algorithmically.

Graph	Polyhedral	$\mathbb{Z}_m$ and D_m	$\mathbb{Z}_r \times \mathbb{Z}_s$ and $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_r \times D_s$ and $D_r \times D_s$
K_7	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, D_3, D_5, D_7	None	None
K_8	A_4, S_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, D_2, D_3, D_4, D_5, D_7, D_8	None	None
K_9	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_9$, D_2, D_3, D_7, D_9	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	None
K_{10}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_9$, $\mathbb{Z}_{10}$, D_2, D_3, D_5, D_7, D_9, D_{10}	None	None
K_{11}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, $\mathbb{Z}_{11}$, D_3, D_5, D_9, D_{11}	None	None
K_{12}	A_4, S_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{12}$, D_2, D_3, D_4, D_5, D_6, D_9, D_{11}, D_{12}	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	None
K_{13}	A_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{13}$, D_2, D_3, D_5, D_{11}, D_{13}	None	None
K_{14}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{13}$, $\mathbb{Z}_{14}$, D_2, D_3, D_7, D_{11}, D_{13}, D_{14}	None	None
K_{15}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_{13}$, $\mathbb{Z}_{15}$, D_3, D_5, D_7, D_{13}, D_{15}	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_2$	None
K_{16}	A_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_{13}$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{16}$, D_2, D_3, D_4, D_5, D_7, D_8, D_{13}, D_{15}, D_{16}	None	None
K_{17}	A_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{17}$, D_2, D_3, D_5, D_7, D_{15}, D_{17}	None	None
K_{18}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{17}$, $\mathbb{Z}_{18}$, D_2, D_3, D_5, D_9, D_{15}, D_{17}, D_{18}	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3 \times D_3$, $D_3 \times D_3$
K_{19}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_9$, $\mathbb{Z}_{17}$, $\mathbb{Z}_{19}$, D_3, D_9, D_{17}, D_{19}	None	None
K_{20}	A_4, S_4, A_5	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{17}$, $\mathbb{Z}_{19}$, $\mathbb{Z}_{20}$, D_2, D_3, D_4, D_5, D_9, D_{10}, D_{17}, D_{19}, D_{20}	None	None

TABLE 1. Possible topological symmetry groups for K_n, when $7 \leq n \leq 20$.

A_4 **Theorem.** [5] *A complete graph* K_m *with* $m \geq 4$ *has an embedding* Γ *in* S^3 *such that* $\mathrm{TSG}_+(\Gamma) \cong A_4$ *if and only if* $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.

A_5 **Theorem.** [5] *A complete graph* K_m *with* $m \geq 4$ *has an embedding* Γ *in* S^3 *such that* $\mathrm{TSG}_+(\Gamma) \cong A_5$ *if and only if* $m \equiv 0, 1, 5, 20 \pmod{60}$.

S_4 **Theorem.** [5] *A complete graph* K_m *with* $m \geq 4$ *has an embedding* Γ *in* S^3 *such that* $\mathrm{TSG}_+(\Gamma) \cong S_4$ *if and only if* $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.

Table 1 lists all the possible topological symmetry groups for embeddings of complete graphs with between 7 and 20 vertices. As a final, more elaborate example, consider K_{140}. In this case the possible topological symmetry groups are:

- A_4, S_4, A_5
- $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{14}$, $\mathbb{Z}_{20}$, $\mathbb{Z}_{23}$, $\mathbb{Z}_{28}$, $\mathbb{Z}_{35}$, $\mathbb{Z}_{69}$, $\mathbb{Z}_{70}$, $\mathbb{Z}_{137}$, $\mathbb{Z}_{139}$, $\mathbb{Z}_{140}$
- D_2, D_3, D_4, D_5, D_7, D_{10}, D_{14}, D_{20}, D_{23}, D_{28}, D_{35}, D_{69}, D_{70}, D_{137}, D_{139}, D_{140}
- $\mathbb{Z}_5 \times \mathbb{Z}_7$, $(\mathbb{Z}_5 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_2$
- $\mathbb{Z}_5 \times D_7$, $\mathbb{Z}_7 \times D_5$, $D_5 \times D_7$

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