

TOPOLOGICAL SYMMETRY GROUPS OF MÖBIUS LADDERS

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ABSTRACT. We classify all groups which can occur as the orientation preserving topological symmetry group of some embedding of a Möbius ladder graph in S^3 .

1. INTRODUCTION

In order to predict molecular behavior, it is useful to know the symmetries of a molecule. The group of rigid molecular symmetries, known as the *point group*, is a convenient way to represent this information. This group is made up of the different rotations, reflections, and combinations of rotations and reflections which preserve a molecular structure in space. However, the larger a molecule is the more flexible it may be. In fact, macromolecules like DNA can be quite flexible. Even some smaller molecules may be partially flexible or contain bonds around which a portion of the molecule can rotate. Such a structure may have a symmetry that is induced by twisting or reflecting just part of the structure, while leaving the rest of the molecule fixed. For example, on the left side of Figure 1 we illustrate a molecular Möbius ladder which is flexible, and on the right side we illustrate a molecule that has rotating propellers on either end. Not all of the symmetries of such molecules are included in the point group.

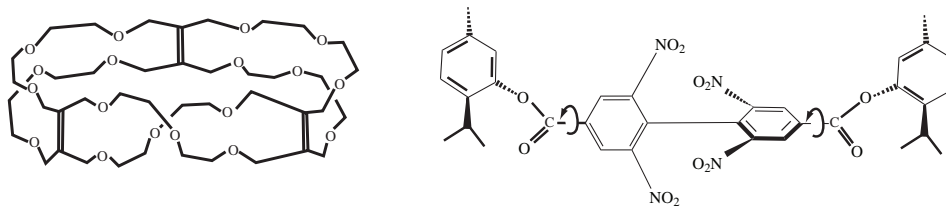


FIGURE 1. The molecule on the left is flexible, while the one on the right has pieces which rotate.

Thus it make sense to represent the symmetries of a non-rigid molecule by the group of automorphisms of the structure that can be induced by homeomorphisms rather than isometries of \mathbb{R}^3 . This group treats molecules as topological rather than geometric structures and hence we call this group of automorphisms the *topological symmetry group*. Note that the topological symmetry group will always be finite

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because it is a subgroup of the automorphism group of a graph, even though the homeomorphisms that induce the automorphisms do not in general have finite order.

In this paper, we are interested in determining the topological symmetry groups of all embeddings of the family of Möbius ladders in \mathbb{R}^3 . The molecular Möbius ladder, synthesized by Walba, Richards, and Haltiwanger [11], has the form of a three-runged ladder with its ends glued together with a half twist so that it resembles the boundary of a Möbius band (as illustrated in Figure 1). The rungs of the ladder are carbon-carbon double bonds, while the sides of the ladder consist of a long polyether chain. This molecule is noteworthy because interest in it led to the development of the interdisciplinary field of topological stereochemistry. In particular, in response to a question by the chemist David Walba who had synthesized the molecule, the topologist Jon Simon [10] proved that there is no orientation reversing homeomorphism of \mathbb{R}^3 taking the molecular Möbius ladder to itself in such a way that rungs go to rungs and sides go to sides. It followed from this result that even though the molecule is somewhat flexible, it must be chemically distinct from its mirror image. Furthermore, Simon showed that no matter how many rungs are added to the molecular Möbius ladder, there would still be no orientation reversing homeomorphism of \mathbb{R}^3 taking the graph to itself.

While the motivation for defining the topological symmetry group came from studying non-rigid molecules, the study of symmetries of graphs in \mathbb{R}^3 is a natural extension of the study of symmetries of knots and links. Furthermore, as is normally done in knot theory, we prefer to embed graphs in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ rather than in \mathbb{R}^3 . Note that an automorphism f of a graph Γ embedded in \mathbb{R}^3 is induced by a homeomorphism of \mathbb{R}^3 if and only if when Γ is considered in $\mathbb{R}^3 \subseteq S^3$ the automorphism f is induced by a homeomorphism of S^3 . Thus the topological symmetry group of an embedded graph is unchanged if we consider the structure in S^3 rather than \mathbb{R}^3 .

In this paper, we consider embeddings of Möbius ladders that do not in general correspond to molecular structures. As an abstract graph, a Möbius ladder with n rungs, denoted by M_n , is a $2n$ -gon with vertices at each corner and edges joining each antipodal pair of vertices. There are infinitely many different ways to embed this graph in S^3 , some containing knots, others containing twists between edges, and most looking nothing like the boundary of a Möbius band. Flapan [2] generalized Simon's result by proving that for every odd $n > 3$ and any embedding Γ of M_n in S^3 , there is no orientation reversing homeomorphism of S^3 taking Γ to itself. On the other hand, for every even n , there is an embedding of Γ in S^3 which has an orientation reversing homeomorphism.

We now consider the question of what groups can occur as topological symmetry groups of some embedding of M_n in S^3 . This question has been addressed for complete graphs K_n in a series of papers [1, 3, 4, 5, 6]. Before we begin we introduce some terminology. In particular, we will refer to a homeomorphism of S^3 taking an embedded graph Γ to itself, as a homeomorphism of the pair (S^3, Γ) . We also need the following definitions.

Definition 1.1. *Let γ be an abstract graph. The group of automorphisms of the vertices of γ is denoted by $\text{Aut}(\gamma)$.*

Definition 1.2. *Let Γ be a graph embedded in S^3 . We define the **orientation preserving topological symmetry group** $\text{TSG}_+(\Gamma)$ as the subgroup of $\text{Aut}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ) .*

Definition 1.3. *Let G be a group and let γ denote an abstract graph. If there is some embedding Γ of γ in S^3 such that $\text{TSG}_+(\Gamma) \cong G$, then we say that the group G is **positively realizable** for γ .*

The goal of this paper is to characterize for each n which groups are positively realizable for M_n . For $n = 1$, the Möbius ladder M_1 is a θ graph which consists of two vertices and three edges going between them. Because there are only two vertices, $\text{Aut}(M_1) \cong \mathbb{Z}_2$. A planar embedding Γ of M_1 , has $\text{TSG}_+(\Gamma) \cong \mathbb{Z}_2$. An embedding Γ of M_1 with a non-invertible knot in one of the edges of M_1 has the trivial group as $\text{TSG}_+(\Gamma)$. This completely characterizes $\text{TSG}_+(\Gamma)$ for embeddings Γ of M_1 . For $n = 2$, observe that $M_2 = K_4$. It was shown in [1] that every subgroup of $\text{Aut}(K_4) \cong S_4$ is positively realizable for K_4 . Thus we focus on M_n for $n \geq 3$.

For $n > 3$, Simon [10] showed that every automorphism of the abstract graph M_n takes the $2n$ -gon to itself. Thus for $n > 3$, the group $\text{Aut}(M_n)$ must be a subgroup of the dihedral group D_{2n} . This makes the analysis of topological symmetry groups of embeddings of M_n much simpler for $n > 3$ than it is for $n = 3$. We consider the topological symmetry groups of embeddings of M_3 in Section 2, and consider the topological symmetry groups of embeddings of M_n for all $n > 3$ in Section 3. Note that any embedding of M_n with $n > 1$ can be modified by adding distinct knots on each edge to obtain an embedding Γ with the trivial group as $\text{TSG}_+(\Gamma)$. Thus from now on we will only consider non-trivial subgroups of $\text{Aut}(M_n)$.

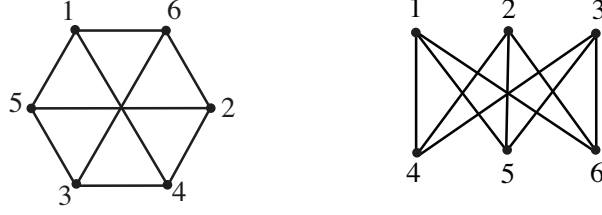
2. TOPOLOGICAL SYMMETRY GROUPS OF EMBEDDINGS OF M_3

In order to obtain a list of groups to check for positive realizability for M_3 , we first observe that M_3 has six vertices and hence its automorphism group is contained in the symmetric group S_6 . Thus we start with the following list of the non-trivial subgroups of S_6 up to isomorphism [1, 9]: $D_6, D_3, D_2, \mathbb{Z}_6, \mathbb{Z}_3, \mathbb{Z}_2, D_3 \times D_3, \mathbb{Z}_3 \times \mathbb{Z}_3, (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2, D_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, S_3 \wr \mathbb{Z}_2$ ¹, $\mathbb{Z}_4, \mathbb{Z}_5, D_4, \mathbb{Z}_2 \times \mathbb{Z}_4, D_4, D_5, A_4, D_4 \times \mathbb{Z}_2, \mathbb{Z}_5 \rtimes \mathbb{Z}_4, A_4 \times \mathbb{Z}_2, S_4, (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4, S_4 \times \mathbb{Z}_2, A_5, S_5, A_6, S_6$.

We can see from Figure 2 that the graph M_3 is equivalent to the complete bipartite graph $K_{3,3}$, which is more convenient to work with. Since the group $\text{Aut}(K_{3,3})$ contains no elements of order 5, we can eliminate many of the groups on the above list. Also, observe that $\text{Aut}(K_{3,3})$ only contains two subgroups of order 3, while A_4 contains four such subgroups. Thus we can also eliminate the groups A_4 and $A_4 \times \mathbb{Z}_2$. Finally note that the automorphism group of $K_{3,3}$ is the same as that of its complementary graph, two triangles. Hence it follows from Frucht [7] that $\text{Aut}(K_{3,3})$ is isomorphic to the wreath product $S_3 \wr \mathbb{Z}_2$ which has order 72.

Now, by a result of Nikkuni and Taniyama [8] the non-trivial automorphisms of $K_{3,3}$ that can be induced by orientation preserving homeomorphisms of some embedding of $K_{3,3}$ in S^3 with vertex sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are contained in the following list, up to conjugation: $(123), (12)(45), (123)(456), (14)(25)(36)$, and (142536) . In particular, no such automorphism has order 4 or is a transposition. It follows that for any embedding Γ of $K_{3,3}$ in S^3 , the group $\text{TSG}_+(\Gamma)$ is isomorphic to a proper subgroup of $\text{Aut}(K_{3,3}) \cong S_3 \wr \mathbb{Z}_2$. Since the order of \mathbb{Z}_2 is 72, this means that the order of any group which is positively realizable for $K_{3,3}$ must be a proper divisor of 72.

¹this is the wreath product which is also denoted by $S_2[S_3]$

FIGURE 2. $K_{3,3}$ and M_3 are the same graph.

Since no transposition of $K_{3,3}$ can be induced by an orientation preserving homeomorphism of (S^3, Γ) for any embedding Γ of $K_{3,3}$ in S^3 , the following lemma implies that the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is not positively realizable for $K_{3,3}$.

Lemma 2.1. *Let $K_{3,3}$ be the complete bipartite graph with vertex sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Let $G \leq \text{Aut}(K_{3,3})$ such that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then G contains a transposition.*

Proof. Suppose that G does not contain a transposition. Since $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, every nontrivial element of G has order 2. Hence every element of G either interchanges the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ or interchanges pairs of vertices in each of the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. The product of two distinct elements of G that interchange the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ is an element of G which interchanges one pair of vertices in each of the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ contains eight involutions, it must contain at least two elements that interchange a pair of vertices in each of the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Thus without loss of generality G contains $(12)(45)$, and either $(12)(46)$ or $(13)(46)$. But this is impossible because then G would contain either $(12)(45)(12)(46) = (456)$ or $(12)(45)(13)(46) = (123)(456)$, both of which have order 3. Thus G must contain a transposition. \square

It now follows that all non-trivial groups that we need to consider for positive realizability for $K_{3,3}$ are contained in the following list, up to isomorphism: D_6 , D_3 , D_2 , \mathbb{Z}_6 , \mathbb{Z}_3 , \mathbb{Z}_2 , $D_3 \times D_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $D_3 \times \mathbb{Z}_3$. This list coincides with the list of all non-trivial subgroups of $\text{Aut}(K_{3,3})$ which contain no order 4 elements or transpositions. We will now show that all of these groups are in fact positively realizable for $K_{3,3}$.

Observe that the group D_6 contains the groups D_3 , D_2 , \mathbb{Z}_6 , \mathbb{Z}_3 , \mathbb{Z}_2 , as subgroups, and the group $D_3 \times D_3$ contains the groups $\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, and $D_3 \times \mathbb{Z}_3$ as subgroups. We deal with these two sets of groups in different subsections. As a matter of convenience, we will refer to the following automorphisms of $K_{3,3}$.

$$\begin{aligned} f &= (123)(456) \\ g &= (123)(465) \\ \psi &= (14)(25)(36) \\ \varphi &= (12)(45) \end{aligned}$$

It is easily checked that the above automorphisms have the following relations: $fg = gf$, $f\psi = \psi f$, $\psi g\psi = g^{-1}$, $\varphi f\varphi = f^{-1}$, $\varphi g\varphi = g^{-1}$, and $\phi(f\psi)\varphi = (f\psi)^{-1}$

2.1. D_6 and its subgroups. In this subsection we create embeddings of $K_{3,3}$ in S^3 to show that the dihedral group D_6 and its subgroups D_3 , D_2 , \mathbb{Z}_6 , \mathbb{Z}_3 , and \mathbb{Z}_2 are positively realizable for $K_{3,3}$. In each case, we will start with a planar embedding of a hexagon with consecutive labeled vertices 1, 6, 2, 4, 3, 5 and stacked edges between antipodal vertices. Then we add knots in various edges to obtain the groups we want. In particular, we use knots to ensure that any homeomorphism of the graph in S^3 will take the hexagon to itself. This will guarantee that the topological symmetry group of each embedding is a subgroup of D_6 .

Note that it was shown in [3] that for 3-connected graphs adding local knots to edges is a well defined operation and that any homeomorphism of S^3 taking the graph to itself must take an edge with a given knot to an edge with the same knot. Furthermore, any orientation preserving homeomorphism of S^3 taking the graph to itself must take an edge with a given non-invertible knot to an edge with the same knot oriented in the same way.

Theorem 2.2. *Every subgroup of D_6 is positively realizable for $K_{3,3}$.*

Proof. We have already shown that the trivial group is realizable, so we only need to consider D_6 and its subgroups D_3 , D_2 , \mathbb{Z}_6 , \mathbb{Z}_3 , and \mathbb{Z}_2 .

We begin with the embedding Γ of $K_{3,3}$ illustrated in Figure 3, in which all of the black squares represent the same invertible knot. Because of these knots any homeomorphism of (S^3, Γ) takes the hexagon to itself, and hence $\text{TSG}_+(\Gamma)$ must be isomorphic to subgroup of D_6 . We define a homeomorphism of (S^3, Γ) by composing a counterclockwise rotation of the hexagon by $2\pi/6$ and a $2\pi/3$ meridional rotation around the hexagon, then isotoping the knots back into position. This homeomorphism induces the automorphism $f\psi = (153426)$. Also a rotation of S^3 pointwise fixing an axis containing the rung $\overline{36}$ takes Γ to itself inducing the automorphism $\varphi = (12)(45)$. Since $\langle f\psi, \varphi \rangle \cong D_6$, we know that $\text{TSG}_+(\Gamma) \cong D_6$.

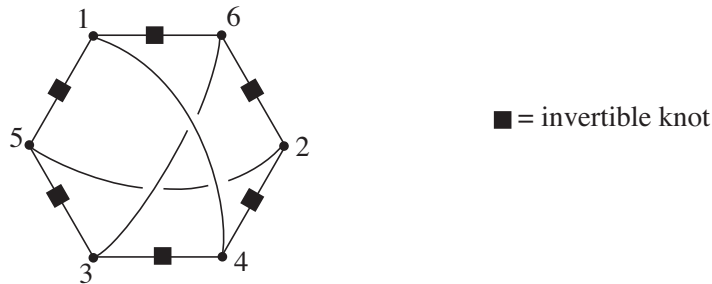
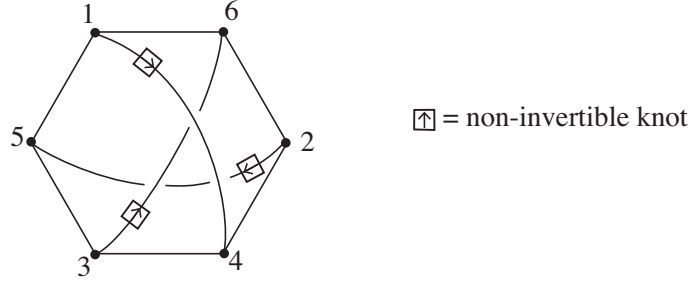


FIGURE 3. $\text{TSG}_+(\Gamma) \cong D_6$.

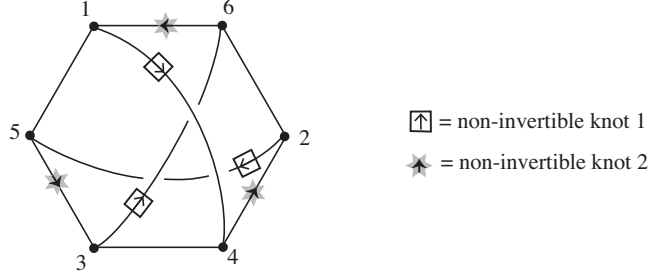
Starting with the embedding Γ in Figure 3, we obtain a new embedding Γ_1 of $K_{3,3}$ by replacing each invertible knot on Γ with equivalent non-invertible knots. The automorphism $f\psi = (153426)$ is induced on Γ_1 by an analogous homeomorphism of (S^3, Γ_1) . However, because of the non-invertible knots there is no orientation preserving homeomorphism of (S^3, Γ_1) which turns the hexagon over. Hence $\text{TSG}_+(\Gamma_1) = \langle f\psi \rangle \cong \mathbb{Z}_6$.

Next let Γ_2 be the embedding of $K_{3,3}$ shown in Figure 4. The boxed arrows represent a particular non-invertible knot which is oriented as indicated. Because

FIGURE 4. $\text{TSG}_+(\Gamma_2) \cong D_3$

of these non-invertible knots the hexagon must be invariant under any homeomorphism of (S^3, Γ_2) , and no even order automorphism of the hexagon can be induced by an orientation preserving homeomorphism. Thus $\text{TSG}_+(\Gamma_2)$ must be isomorphic to subgroup of D_3 . Now the automorphism φ is induced by a rotation of (S^3, Γ_2) pointwise fixing an axis containing the edge $\overline{36}$. We define a homeomorphism h_2 of (S^3, Γ_2) by composing a clockwise $2\pi/3$ rotation and a $2\pi/3$ rotation meridionally around the hexagon. Then h_2 induces the automorphism $f = (123)(456)$ of Γ_2 . Therefore, $\text{TSG}_+(\Gamma_2) = \langle f, \varphi \rangle \cong D_3$.

Starting with the embedding Γ_2 illustrated in Figure 4, we now add noninvertible knots to every other edge of the hexagon to obtain a new embedding Γ_3 of $K_{3,3}$ as shown in Figure 5. Because the knots on the hexagon are noninvertible, no orientation preserving homeomorphism of (S^3, Γ_3) turns the hexagon over. But we can still induce the automorphism $f = (123)(456)$. Thus, in this case, $\text{TSG}_+(\Gamma_3) = \langle f \rangle \cong \mathbb{Z}_3$.

FIGURE 5. $\text{TSG}_+(\Gamma_3) \cong \mathbb{Z}_3$

Now, consider the embedding Γ_4 of $K_{3,3}$ in Figure 6 which has equivalent invertible knots in edges $\overline{15}$ and $\overline{24}$ and a different invertible knot in the remaining edges of the hexagon.

Any homeomorphism of (S^3, Γ_4) must take the hexagon to itself either setwise fixing both $\overline{15}$ and $\overline{24}$ or interchanging them. Rotating the hexagon by 180° induces the automorphism $\psi = (14)(25)(36)$, and turning the graph about an axis containing the edge $\overline{36}$ induces φ . The product $\psi\varphi = (15)(24)(36)$ is the only

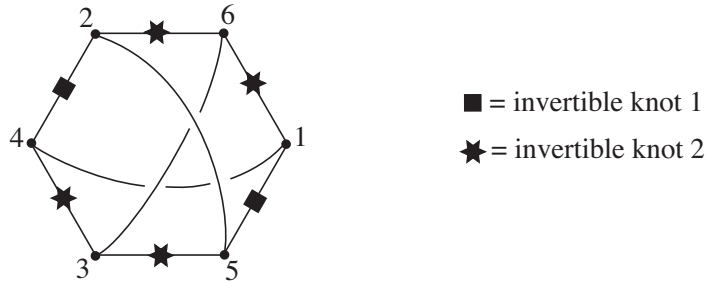


FIGURE 6. $\text{TSG}_+(\Gamma_4) \cong D_2$

nontrivial automorphism which setwise fixes both $\overline{15}$ and $\overline{24}$. Thus, $\text{TSG}_+(\Gamma_4) = \langle \psi, \varphi \rangle \cong D_2$.

Finally, consider the embedding Γ_5 of $K_{3,3}$ shown in Figure 7 with four distinct invertible knots on the edges of the hexagon as indicated. The homeomorphism of S^3 which turns the graph over about an axis through edges $\overline{14}$ and $\overline{25}$ induces the automorphism $\psi = (14)(25)(36)$ on Γ_5 . Since $\overline{14}$ and $\overline{25}$ must each be setwise fixed by any homeomorphism of (S^3, Γ_5) , the automorphism ψ is the only nontrivial element in $\text{TSG}_+(\Gamma_5)$. Thus, $\text{TSG}_+(\Gamma_5) = \langle \psi \rangle \cong \mathbb{Z}_2$. \square

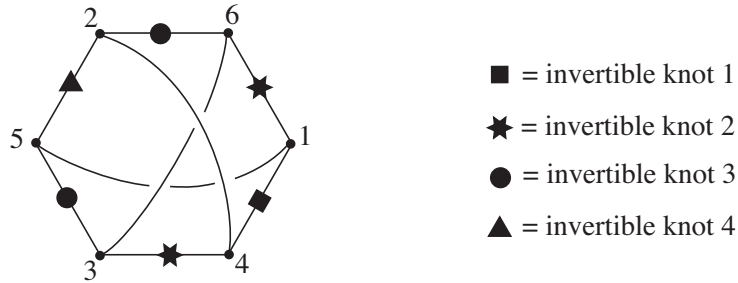


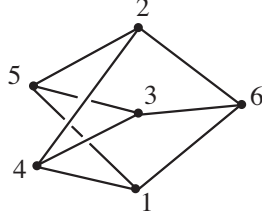
FIGURE 7. $\text{TSG}_+(\Gamma_5) \cong \mathbb{Z}_2$

2.2. $D_3 \times D_3$ and its subgroups. In this subsection we create embeddings of $K_{3,3}$ in S^3 to show that the group $D_3 \times D_3$ and its subgroups $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $D_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are positively realizable for $K_{3,3}$. In particular, we prove the following theorem.

Theorem 2.3. *Every subgroup of $D_3 \times D_3$ is positively realizable for $K_{3,3}$*

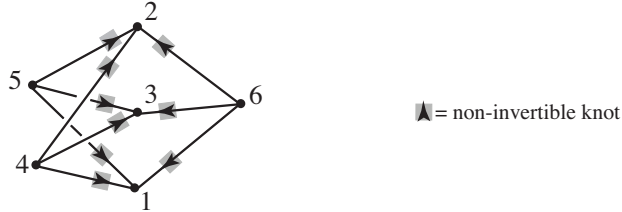
Proof. Since we proved that all of the subgroups of D_6 are positively realizable in Theorem 3, here we only need to prove that the group $D_3 \times D_3$ and its subgroups $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $D_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are positively realizable for $K_{3,3}$

We begin with an embedding Γ of $K_{3,3}$ in the form of a fan with three blades as shown in Figure 8. We can think of S^3 as the union of two solid tori with the core of one solid torus containing the vertices 1, 2, and 3, and the core of the other solid

FIGURE 8. $\text{TSG}_+(\Gamma) \cong D_3 \times D_3$.

torus containing vertices 4, 5, and 6. The automorphisms f and g are induced on Γ by order 3 rotations about the cores of both solid tori. Turning over the graph about an axis containing the edge $\overline{36}$ induces the automorphism φ on Γ . Finally, the homeomorphism of S^3 which interchanges the two cores induces $\psi = (14)(25)(36)$. Therefore, $\langle f, \varphi, g, \psi \rangle \cong D_3 \times D_3 \leq \text{TSG}_+(\Gamma)$. However, since the order of any positively realizable group for $K_{3,3}$ must be a proper divisor of 72 and the order of $D_3 \times D_3$ is 36, we know that in fact $\text{TSG}_+(\Gamma) \cong D_3 \times D_3$.

Now, we modify the fan embedding Γ illustrated in Figure 8 by placing a non-invertible knot in each edge, giving us a new embedding Γ_1 of $K_{3,3}$ shown in Figure 9. Observe that the non-invertible knots are oriented from vertices 4, 5, and 6 towards vertices 1, 2, and 3. Hence no orientation preserving homeomorphism of (S^3, Γ_1) can interchange the set of vertices $\{1, 2, 3\}$ with the set of vertices $\{4, 5, 6\}$. Also, we know that there is no transposition in $\text{TSG}_+(\Gamma_1)$. On the other hand, the automorphisms f, g , and φ can be induced by homeomorphisms of (S^3, Γ_1) as they were for (S^3, Γ) . Recall that we have the relations $\varphi f \varphi = f^{-1}$ and $\varphi g \varphi = g^{-1}$. Thus $\text{TSG}_+(\Gamma_1) = \langle f, g, \varphi \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

FIGURE 9. $\text{TSG}_+(\Gamma_1) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

In order to find an embedding Γ_2 of $K_{3,3}$ such that $\text{TSG}_+(\Gamma_2) \cong D_3 \times \mathbb{Z}_3$, we again start with the embedding Γ illustrated in Figure 8. Now in a small neighborhood of each vertex $x \in \{1, 2, 3\}$ and $a \in \{4, 5, 6\}$ we embed the arcs of the edges as illustrated in Figure 10. This gives us an embedding Γ_2 of $K_{3,3}$.

If a local knot in an edge e_1 is linked with an edge e_2 (as $\overline{6x}$ is with $\overline{4x}$ in Figure 10) we will say that e_1 is *knotted around* e_2 . Thus for each vertex $x \in \{1, 2, 3\}$, the edge $\overline{x4}$ is knotted around the edge $\overline{x5}$, which in turn is knotted around the edge $\overline{x6}$, which in turn is knotted around the edge $\overline{x4}$. Similarly, for each vertex $a \in \{4, 5, 6\}$, the edge $\overline{a1}$ is knotted around the edge $\overline{a2}$, which in turn

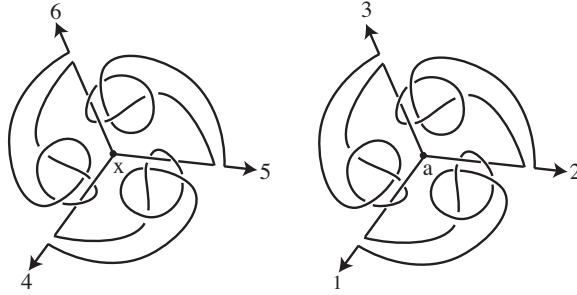


FIGURE 10. These knots orient the three edges around each vertex.

is knotted around the edge $\overline{a3}$, which in turn is knotted around the edge $\overline{a1}$. On the other hand, none of the reverse knotting around relations hold. Suppose that there were some homeomorphism of (S^3, Γ_2) that interchanged pairs of vertices in each of the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Without loss of generality, suppose that some homeomorphism of (S^3, Γ_2) induces the automorphism $\varphi = (12)(45)$. Thus the edges $\overline{14}$ and $\overline{15}$, are mapped to the edges $\overline{25}$ and $\overline{24}$ respectively. But this is impossible because the edge $\overline{14}$ is knotted around the edge $\overline{15}$, while the edge $\overline{25}$ is not knotted around the edge $\overline{24}$. Thus no homeomorphism of (S^3, Γ_2) interchanges pairs of vertices within the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

The automorphisms f and g are induced on Γ_2 by order 3 rotations about the cores of both solid tori as they were induced on Γ . Figure 11 illustrates an isotopy of S^3 inducing the automorphism $\psi = (14)(25)(36)$ on the embedding Γ while preserving the orientation of the edges around each vertex. A similar isotopy induces ψ on Γ_2 preserving the knots around the edges. Thus $\psi \in \text{TSG}_+(\Gamma_2)$. It follows that $\text{TSG}_+(\Gamma_2) = \langle f, g, \psi \rangle \cong D_3 \times \mathbb{Z}_3$.

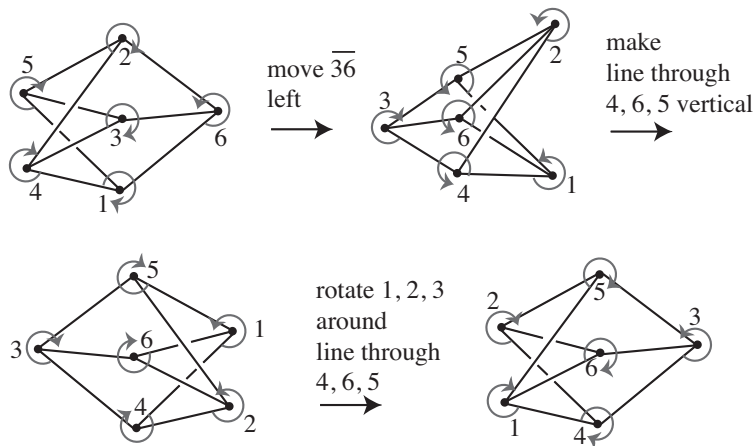


FIGURE 11. This isotopy induces ψ on Γ preserving the orientation of the edges around each vertex.

Finally, consider the embedding Γ_3 obtained from the embedding Γ_2 by adding a non-invertible knot to each edge oriented from the vertices 4, 5, and 6 towards the vertices 1, 2, and 3 as they were in Figure 9. The automorphisms f and g are induced by homeomorphisms of (S^3, Γ_3) . However, now there is no homeomorphism of (S^3, Γ_3) which interchanges the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Thus $\text{TSG}_+(\Gamma_3) = \langle f, g \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. \square

The following Corollary is a summary of our results for M_3 .

Corollary 2.4. *All subgroups of $\text{Aut}(K_{3,3})$ containing no element of order 4 and no transposition are positively realizable for M_3 .*

3. EMBEDDINGS OF M_n FOR $n > 3$.

Simon [10] proved that for $n \geq 4$, every automorphism of M_n , leaves the $2n$ -gon setwise invariant. Thus, for $n \geq 4$, $\text{Aut}(M_n) \cong D_{2n}$, the dihedral group of order $4n$. Note that every nontrivial subgroup of D_{2n} is isomorphic to D_k or \mathbb{Z}_k , where k divides $2n$. We will now prove that each of these dihedral and cyclic groups is positively realizable for M_n .

Theorem 3.1. *For all $n \geq 4$, every subgroup of D_{2n} is positively realizable for M_n .*

Proof. Consider the embedding of M_4 shown in Figure 12. We can obtain a similar embedding for any M_5 by adding a vertex between vertices 1 and 8 and a vertex between vertices 4 and 5, and a new rung that goes above $\overline{15}$. By repeating this process we obtain a similar embedding Γ of M_n for any $n \geq 4$.

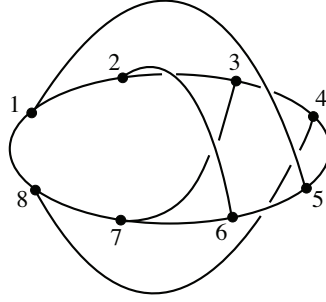


FIGURE 12. $\text{TSG}_+(\Gamma) = D_{2n}$

Let α be a homeomorphism of (S^3, Γ) which turns Γ over. Note that α pointwise fixes an edge precisely when n is odd. Let β be a glide rotation of S^3 which rotates the $2n$ -gon by $\frac{\pi}{n}$ and rotates meridionally around the $2n$ -gon by $\frac{2\pi}{n}$ taking each rung to the next lower rung. Then β takes Γ to itself inducing an order $2n$ automorphism on Γ . It follows that $\text{TSG}_+(\Gamma) = D_{2n}$.

Next we let Ω_1 be obtained from Γ by adding the same non-invertible knot to every edge of the $2n$ -gon. Let β_1 be the homeomorphism of (S^3, Ω_1) which rotates the $2n$ -gon by $\frac{\pi}{n}$, rotates meridionally around the $2n$ -gon by $\frac{2\pi}{n}$, and then isotopes the knots back into position. Because of the non-invertible knots, the $2n$ -gon cannot be turned over. Thus, $\text{TSG}_+(\Omega_1) \cong \mathbb{Z}_{2n}$.

In general, let k be a nontrivial divisor of $2n$, and let $m = 2n/k$. Let e_1, \dots, e_{2n} denote consecutive edges that make up the $2n$ -gon. Starting with the embedding Γ add equivalent non-invertible knots to each of the edges $e_1, e_{1+m}, \dots, e_{2n-m}$ to obtain an embedding Ω_m of M_n . Let β_m be the homeomorphism of (S^3, Ω_m) which rotates the $2n$ -gon by $\frac{m\pi}{n}$, rotates meridionally around the $2n$ -gon by $\frac{2m\pi}{n}$, and then isotopes the knots back into position. Then β_m will take Ω_m to itself and induce an automorphism of M_n of order k . Now, because the knots are non-invertible, no homeomorphism of (S^3, Ω_m) will turn the $2n$ -gon over. Also because of the knots, any homeomorphism of (S^3, Ω_m) induces a rotation of the $2n$ -gon whose order is a factor of k . Thus, $\text{TSG}_+(\Omega_m) \cong \mathbb{Z}_k$.

Finally, let Γ_m be obtained from the embedding Ω_m by replacing the non-invertible knots in Ω_m with equivalent invertible knots. Then there is a homeomorphism of (S^3, Γ_m) inducing an automorphism of M_n of order k , and there is a homeomorphism of (S^3, Γ_m) which turns the $2n$ -gon over. Thus $\text{TSG}_+(\Omega_m) \cong D_k$. \square

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