BASIC FACTS ABOUT WEAKLY SYMMETRIC SPACES

GIZEM KARAALI

1. Symmetric Spaces

Symmetric spaces in the sense of E. Cartan supply major examples in Riemannian geometry. The study of their structure is connected closely to the theory of Lie groups. On these spaces, global analysis, particularly integration theory and theory of partial differential operators, arises in a canonical fashion by the requirement of geometric invariance. On R^n these two subjects are related to one another by the Fourier transform.

Here we give the basic definition and facts about symmetric spaces; one can check [1] for a more detailed introduction to this topic:

A Riemannian manifold M is called **symmetric** if for every $p \in M$, there exists an isometry $\sigma_p : M \to M$ with

$$\sigma_p(p) = p \text{ and } d\sigma_p(p) = -id$$

Such isometries are also called **involution**s.

Basic examples are CP^n with the Fubini-Study metric, R^d equipped with the Euclidean metric, the sphere S^d, etc. One can show that symmetric spaces are geodesically complete and this helps one to prove that the involutive isometries in the definition of the symmetric space are actually uniquely determined. All symmetric spaces are homogeneous, ie they have a transitive group of isometries, and the isometry group of a symmetric space is a Lie group (and so here is the connection to Lie groups!!). Symmetric spaces can be completely classified into a fnite number of series plus a finite list of exceptional ones. Harmonic analysis on compact symmetric spaces is well-developed through the Peter-Weyl theorem for compact groups and Cartan's further developments of this. Helgason's monograph [2] is devoted to geometric analysis on noncompact Riemannian symmetric spaces.

2. Weakly Symmetric Spaces

Now let's assume that M is a (Riemannian) homogeneous space isomorphic to G/K. Let u be a fixed isometry of M (where u is not necessarily in G) such that $uGu^{-1} = G$ and $u^2 \in G$, and given any two points $x, y \in M$ there is $g \in G$ such that $g \cdot x = u \cdot y$ and $g \cdot y = u \cdot x$. Then (M, G, u) – or just M – is called **weakly symmetric**. This definition is due to A. Selberg [3].

[There are alternate characterizations (by Berndt and Van Hecke) of the notion of "weakly symmetric", based on the reversal of geodesics and in this way one can see a more geometric relation to the notion of "symmetric space". (Check references of [5]) Also Prof Wolf will talk about Ziller's paper where still another characterization is provided.]

As symmetric spaces are only a special case, symmetric spaces form a large class of examples. [If M is symmetric, and $x, y \in M$, then as M is geodesically complete, one can find a shortest geodesic connecting x to y. Let p be the "midpoint" of this geodesic. Then the isometry σ_p will satisfy $\sigma_p(x) = y$, and $\sigma_p(y) = x$. Thus $(M, I(M), id)$ is a weakly symmetric space, where $I(M)$ is the group of isometries of M]

Of course one needs to find some weakly symmetric examples which are not symmetric. (Otherwise we would need to look for a theorem of equivalence between the two definitions!). Here is the example Prof. Wolf suggested (and I do not claim to have understood this example):

Consider an irreducible bounded symmetric domain G/K, (for instance let $G = Sp(n; R)$ and $K = U(n)$). Assume that G is simple, and the maximal compact K is not simple but is actually of the form $[K, K] \cdot {Z_K}^0$, (note here that $[K, K]$ is semisimple, and ${Z_K}^0$, the identity component of the center of K, is a circle). (In the above example, we see that $U(n) = SU(n) \cdot \{e^{i\theta} I_n\}$, and the second factor here is the central circle which gives the complex structure to $U(n)$). Now consider the circle bundle $G/[K, K]$ over G/K. Then if $D = G/K$ is not a *tube domain*, then $G/[K, K]$ is weakly symmetric, but as $[K, K]$ is not a maximal compact, it is not symmetric. For our example, the above gives the total space of principal circle bundles over the Siegel half-space with a twisted action by the group $Sp(n; R) \times S^1$.

3. Why Are We Interested in Weakly Symmetric Spaces?

Recall the definition of a Gelfand pair: Let G be a locally compact group and K a compact subgroup. Then (G, K) is a **Gelfand pair** if the convolution algebra ${L^1}_{K,K}(G)$ of L^1 functions which are both left- and right-K-invariant, is commutative. (Recall that a function f on G is left-K-invariant if $f(kg) = f(g)$ for all $g \in G, k \in K$. This means that it is constant on each left coset. Such functions correspond to functions f' on G/K, via $f = f' \emptyset \pi_K$ where π_K is the usual projection $G \to G/K$. Right-invariance is defined in a similar way. So actually we are interested in functions on the double coset space of G with respect to K, $L^1(K \diagdown G \diagup K)$)

The definition we gave above for a weakly symmetric space is due to A. Selberg [3]. There, Selberg is interested in spaces M for which the algebra of G-invariant linear operators is commutative. So here is our connection to Gelfand pairs: Selberg's weakly symmetric spaces provide us with examples of Gelfand pairs. To be more precise, Selberg in [3] proves that the algebra of G-invariant differential operators on $M = G/K$ is commutative, when (M, G, u) is weakly symmetric. [We will try and explain Selberg's proof of this fact in the following section.] Then we combine this with Thomas's result in [4] (which will be described in section 5) that under the assumption that G is connected, the commutativity of this algebra is equivalent to (G, K) being a Gelfand pair, and we see that weakly symmetric spaces give us a good set of examples for Gelfand pairs.

4. Commutative Spaces of G-invariant Operators, Selberg's Proof

[This section is a recap of the first few pages of Selberg's [3].]

Let M be a Riemannian manifold. Let x denote a point of M, and let $x_1, \ldots, x_n$ be the corresponding local coordinates. Let G be a locally compact group of isometries of M which acts transitively, and assume that (M, G, u) is a weakly symmetric space. Let $\mathcal{S}$ be the space of linear operators (on the functions on M) that commute with all the isometries in G which will be further restricted to include only the differential operators of finite order, integral operators of the form $\int_M k(x, y) f(y) dy$ (where dy denotes the G-nvariant volume form), or any finite combination (by addition or multiplication) of such.

[This is the algebra that Selberg proves is commutative. This is actually more than what we need; we only need the algebra of G-invariant differential operators. Fixing a base point say $x_0 \in M$, and letting K be the compact isotropy subgroup of G of x_0, we have that $M = G/K$ and $\mathcal{S}$ contains the algebra of G-invariant differential operators on G/K.]

Selberg first notes that requiring an integral operator of the form $\int_M k(x,y)f(y)dy$, to be G-invariant is equivalent to requiring that the kernel satisfy $k(g \cdot x, g \cdot y) = k(x,y)$ for all $x, y \in M$ and $g \in G$. He names such kernels **point-pair invariant**. If we consider such a "point-pair invariant" $k(x,y)$ as a function of x only, fixing y, we see that $k(x,y)$ is invariant under the subgroup of G that fixes y. (Selberg calls this subgroup the **rotation group** of y and denotes it by Ry.) In this case we say that k has, as a function of x, rotational symmetry around the point y. Selberg then moves on to show that the study of point-pair invariants is equivalent to the study of functions with rotational symmetry around some fixed point x_0.

Selberg also notes that as the G action is transitive, an invariant operator in $\mathcal{S}$ is completely determined by its action at one point. In the same spirit, starting with any linear operator (not necessarily G-invariant) and any point x_0 of M, we can construct a G-invariant operator that agrees with the original operator at x_0, provided that the original operator is invariant under the rotation group of the point x_0. (If we have an operator which is not even this nice, ie is not stable under the rotation group of any point x_0 then we still can define a G-invariant operator starting with this operator by using the symmetrized function corresponding to the function we are looking at at the chosen point.. The construction of the symmetrized function for any given function f and any given point x_0 will be described later in this section..) Furthermore one can see that a G-invariant operator applied to a function with rotational symmetry around a point, gives a function which is rotationally symmetric around the same point. As a result, a G-invariant operator applied to a point-pair invariant (considered as a function of say the first point) gives again a point-pair invariant.

Consider the G-invariant differential operators in $\mathcal{S}$ now. (Recall that this is the class of operators we are more interested in). Pick local coordinates around x_0 such that the matrix $(g_{ij}(x_0))$ is the identity matrix. The action of any G-invariant differential operator of finite order at the point x_0 is identical with the action of a differential operator of finite order with constant coefficients. Using this fact, Selberg then sets up a correspondence between homogeneous polynomials (of n variables) invariant under the group of rotations of x_0, (to be more precise the group of rotations of the tangent space at x_0 that is determined by Rx_0), and G-invariant differential operators of finite order. Then (by a theorem of Hilbert which says that we can find a finite basis of homogeneous polynomials for the space of polynomials such that every other homogeneous polynomial can be written as a polynomial in these) we get the result that the corresponding differential operators generate the ring of the G-invariant differential operators (in the sense that any other such can be written as a finite polynomial in these chosen ones). Selberg calls these generators **fundamental operators**.

Finally we introduce the following notation: $[f(x)]_{x \to x_0}$. This is supposed to stand for the value of the function $f(x)$ at the point x_0. This then is the setup of [3]. Now we will look at the proof of the fact that $\mathcal{S}$ is commutative.

We first see that all elements of $\mathcal{S}$ commute when applied to point-pair invariants $k(x, y)$ considered as functions of the first point x. First note that if L is an invariant operator then so is $\overline{L}$ where $\overline{L}$ is defined as:

$$\begin{aligned}\overline{L}f(x) = &\text{ the value of the function } Lf(u^{-1} \cdot x) \text{ at the point } u \cdot x \\ &= [Lf(u^{-1} \cdot x)]_{x \to u \cdot x}\end{aligned}$$

Also from our assumption about u we have, for any point-pair invariant $k(x, y)$:

$$k(u \cdot y, u \cdot x) = k(g \cdot x, g \cdot y) = k(x, y)$$

Denoting by a subscript the argument that the operator is to act on we have:

$$L_x k(x, y) = k'(x, y)$$

where $k'(x, y)$ is again a point-pair invariant. We have then:

$$\begin{aligned}\overline{L}_y k(x, y) &= \overline{L}_y k(u \cdot y, u \cdot x) = [L_y k(y, u \cdot x)]_{y \to u \cdot y} \\ &= [k'(y, u \cdot x)]_{y \to u \cdot y} = k'(u \cdot y, u \cdot x) = k'(x, y)\end{aligned}$$

which finally gives us:

$$L_x k(x, y) = \overline{L}_y k(x, y)$$

so we may actually shift the operator from the first to the second argument by going from L to $\overline{L}$. If we now have two operators $L^{(1)}$ and $L^{(2)}$, we can see that:

$$L_x^{(1)} L_x^{(2)} k(x,y) = L_x^{(1)} \overline{L^{(2)}}_y k(x,y) = \overline{L^{(2)}}_y L_x^{(1)} k(x,y) = L_x^{(2)} L_x^{(1)} k(x,y)$$

since the operators acting on different arguments commute. So we have commutativity when our operators are applied to point-pair invariants. So we have comutativity if our operators are applied to a function with rotational symmetry around a point, say x_0. So finally let f be a function without any rotational symmetry. Then we define the corresponding symmetrized function $f(x : x_0)$ as:

$$f(x : x_0) = \int_{Rx_0} f(r \cdot x) dr$$

where dr corresponds to the normalized bi-invariant Haar measure on Rx_0. This function has rotational symmetry around x_0 and for any two invariant operators $L^{(1)}, L^{(2)}$, we see that the value at x_0 of the function $L^{(1)} L^{(2)} f(x)$ is equal to the value at (the same point) x_0 of the function $L^{(1)} L^{(2)} f(x : x_0)$. But then from here we conclude that $L^{(1)} L^{(2)} f(x) = L^{(2)} L^{(1)} f(x)$, because this holds for functions with symmetry.. This completes the proof; all the operators in $\mathcal{S}$ commute.

[One last observation: Selberg also notes that the complex conjugate of $\overline{L}$ is the formal adjoint of the operator L.]

5. More on Gelfand Pairs, Commutative Spaces, Thomas's Proof

Here we will try to describe the result of Thomas [4] that Gelfand pair = commutative space. This is done by a Lie algebra calculation, so the big group G has to be connected. [Later Prof. Wolf will describe Nguyen's work [5] where one can drop this connectedness hypothesis.] More precisely we are trying to prove the following theorem:

Theorem 1. *Let G be a connected Lie group, K a compact subgroup. Then (G, K) is a Gelfand pair if and only if the algebra of G-invariant differenetial operators on G/K is commutative.*

Let's define the spaces involved first:
$\mathcal{D}(G)$ = the space of smooth functions on G with compact support [the dual of this space is the space $\mathcal{D}'(G)$, the space of distributions]
$C^c(G)$ = the space of continuos functions on G with compact support
$\mathcal{E}'(G)$ = the space of compactly supported distributions on G [this is

the dual space for the space $\mathcal{E}(G)$ which is the space of all smooth functions on G]

For $\mathcal{H}$ any of these above spaces, let $\mathcal{H}_{K,K}$ denote the subspace of functions or distributions which are both left- and right-K-invariant. Again for $\mathcal{H}$ any of these above spaces, let $\mathcal{H}_{;K}$ denote the right-K-invariant elements in $\mathcal{H}$. And finally let δ_K denote the normalized left Haar measure on K amd δ_x denote the Dirac measure of $x \in G$ for any $x \in G$.

Now we are ready to start the proof of our theorem. First we assume we have a Gelfand pair (G, K) and try to prove commutativity. Then following Thomas in [4] we have:

Proposition 1. *Let $S \in \mathcal{E}'_{K,K}(G)$ (where the latter is the space of compactly supported distributions on G which are both left- and right-K-invariant) and A_S be the linear operator in $\mathcal{D}'_{;K}(G)$ defined by $A_S(T) = T \star S$. (Here the space $\mathcal{D}'_{;K}(G)$ is the space of distributions on G which are right-K-invariant). Then A_S is continuous and commutes with left translations. Conversely every continuous linear operator A in $\mathcal{D}'_{;K}(G)$ which commutes with left translations is of the form A_S for a unique $S \in \mathcal{E}'_{K,K}(G)$. Moreover $A_{S_1 \star S_2} = A_{S_2} A_{S_1}$.*

[One remark about the proof: The easiest part is the last result, which follows from associativity of the convolution product $\star$.]

Proposition 2. *Consider the following convolution algebras:*
$\mathcal{D}_{K,K}(G)$ = the space of smooth functions on G with compact support that are both left- and right-K-invariant
$C^c{}_{K,K}(G)$ = the space of contionuos functions on G with compact support that are both left- and right-K-invariant
$L^1{}_{K,K}(G)$ = the space of L^1 functions on G which are both left- and right-K-invariant
$\mathcal{E}'_{K,K}(G)$ = the space of compactly supported distributions on G which are both left- and right-K-invariant
Then if one of these is commutative, so are the others.

[If $\mathcal{H}$ is any one of the spaces $\mathcal{D}(G), C^c(G), L^1(G), \mathcal{E}'(G)$, the operator $P_{K,K}$ defined by $P_{K,K}(T) = \delta_K \star T \star \delta_K$ is a continuous operator from $\mathcal{H}$ onto $\mathcal{H}_{K,K}$. But $\mathcal{D}(G)$ is dense in $C^c(G), L^1(G), \mathcal{E}'(G)$, and so we get that $\mathcal{D}_{K,K}(G)$ is dense in the above spaces of the propostion.]

Combining these two propositions and the fact that the space of distributions on G/K (denoted by $\mathcal{D}'(G/K)$) is isomorphic to $\mathcal{D}'_{;K}(G)$ we get:

Proposition 3. *Let G be a Lie group and K a compact subgroup. Then (G, K) is a Gelfand pair if and only if the algebra of continuous linear operators $A : \mathcal{D}'(G/K) \to \mathcal{D}'(G/K)$ which commute with the G action is a commutative algebra.*

[This algebra in this proposition corresponds, via Prop. 1, to $\mathcal{E}'_{K,K}(G)$, and its commutativity by Prop. 2, is equivalent to the commutativity of $L^1{}_{K,K}(G)$]

In particular one can see that if (G, K) is a Gelfand pair, then the algebra of G-invariant differential operators $D : \mathcal{D}'(G/K) \to \mathcal{D}'(G/K)$ is commutative. [This is a subalgebra of the algebra considered in Prop. 3.]

[Note that the harder part of the proof is still to come. We have now to prove that if we have that the algebra of G-invariant differential operators on G/K is commutative then (G, K) is a Gelfand pair. This is the result one needs to finish the proof that the weakly symmetric spaces give us Gelfand pairs. However recall again that here we have to assume that G is connected..]

Now to prove the remaining direction we will first look at the subalgebra of $\mathcal{E}'_{K,K}$ corresponding to the invariant differential operators on G/K. Let $\mathcal{A}$ be the space of distributions S of the form $S = \delta_K \star E \star \delta_K$ where E is a distribution supported at $\{e\}$. Thomas then proves here the following:

Proposition 4. *Let $S \in \mathcal{E}'_{K,K}$. Then the operator A_S of Proposition 1 (ie $A_S : \mathcal{D}'_{;K}(G) \to \mathcal{D}'_{;K}(G)$ defined by $A_S(T) = T \star S$) corresponds to a differential operator on G/K if and only if $S \in \mathcal{A}$.*

[Here we do not include the proof. Although it is not very short, this proof is rather easy to follow. Check [4]]

Then we need to introduce $C^\omega(G)$ the space of real-anlytic functions $f : G \to C$ on G. then Thomas's fifth proposition is the well known fact that $C^\omega(G)$ is a dense subspace of $\mathcal{E}(G)$, which is just the space of smooth functions on G, (this is the space whose dual is $\mathcal{E}'(G)$, the space of distributuions with compact support..) Thomas actually provides two nice proofs of this fact but here we will not go into these proofs.. [Actually here Thomas goes deeper into the topologies of the underlying spaces, and density is with respect to the topology he denotes by $\sigma(\mathcal{E}'(G), C^\omega(G))$. This is the weakest topology for which the maps $T \mapsto T(f)$, where $T \in \mathcal{E}'(G)$ and $f \in C^\omega(G)$, are continuous.]

Here we carry on with the proof of the main theorem assuming this fact. Thus we skip to the following proposition which appears as Proposition 8 in Thomas:

Proposition 5. *Let G be a connected Lie group. Then the space of distributions on G supported at $\{e\}$ is dense in $\mathcal{E}'(G)$, where the latter is equipped with the topology $\sigma(\mathcal{E}'(G), C^\omega(G))$.*

This is a result due to Godement (references in [4]), but it is actually not too hard to give a sketch of the proof at this stage: If $f \in C^\omega(G))$ and $T(f) = 0$ for all distributions T on G with $supp(T) = \{e\}$, then we have that $Df(e) = 0$ for all left- ot right-invariant differential operators D, and so in particular, we get $f = 0$ [Thomas's note: In the above application of Hahn-Banach theorem, the spaces are all separable, so only countable choice is involved. Note to self: Here is the version of Hahn-Banach theorem we are using: Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then $x_0 \in \overline{M}$ if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for all $x \in X$ but $f(x_0) \neq 0$. Our linear functional is $T \mapsto T(f)$.]

Here is the last proposition of Thomas, it appears as the ninth in [4]:

Proposition 6. *Let $S \in \mathcal{E}'(G)$. Then the operators $T \mapsto T \star S$ and $T \mapsto S \star T$ are continuous in $\mathcal{E}'(G)$ if this space is equipped with the topology $\sigma(\mathcal{E}'(G), C^\omega(G))$.*

Finally we get to the end of the proof of the main theorem. By the last proposition (Prop. 9 of [4], here it is Prop. 6), we know that the operator $P_{K,K} : \mathcal{E}'(G) \to \mathcal{E}'(G)$ defined by $P_{K,K}(T) = \delta_K \star T \star \delta_K$ is continuous when $\mathcal{E}'(G)$ is equipped with the topology $\sigma(\mathcal{E}'(G), C^\omega(G))$. By Prop. 4, we know that the image $\mathcal{A} = P_{K,K}$(distributions supported at $\{e\}$) is the subalgebra of $\mathcal{E}'_{K,K}$ corresponding to the invariant differential operators on G/K. Thus by Prop. 5 (Prop. 8 of [4]) $\mathcal{A}$ is dense in $\mathcal{E}'_{K,K}(G)$, where the latter is equipped with the topology induced by $\sigma(\mathcal{E}'(G), C^\omega(G))$ [Recall that the operator $P_{K,K}$ is continuous and it maps $\mathcal{E}'(G)$ onto $\mathcal{E}'_{K,K}(G)$. So as $\mathcal{A}$ is the image of a dense subset under a continuous surjective map, it is dense in the whole image.] Therefore if $\mathcal{A}$ is commutative, then it follows [Thomas refers here to the last proposition (Prop. 9 of [4], here it is Prop. 6), I do not exactly see the need] that $\mathcal{E}'_{K,K}(G)$ is commutative, and finally by Prop. 2, this implies that (G, K) is a Gelfand pair.

6. Bibliography:

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