Four Colors is not Enough: Visualizations of Simulated Spatial-Model Elections Under Different Voting Methods

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Chapter 1

Introduction

1.1 History of Voting Theory

The modern study of voting methods was born out of an 18th century debate over how to best elect members to the French Academy of Science. Jean-Charles de Borda proposed a now eponymous voting method for electing members to the academy - each voter ranks the candidates in order with the top choice receiving as many points as there are candidates ranked below them. Borda's was adopted by the academy, but in 1785 it was subjected to a powerful critique by the Marquis de Condorcet who constructed a convincing example of its apparently undesirably behavior. Condorcet considered what would happen if each possible pairing of candidates in an election were considered separately. In Condorcet's example there was a candidate that beat all other candidates in head-to-head match ups but failed to win under Borda's Method.

Since then the Condorcet criterion has remained a popular standard for measuring the desirability and behavior of voting methods. The search for an ideal voting method continued until 1950 when Arrow proved his possibility theorem, better known as Arrow's Impossibility Theorem. [Arr50] Arrow's approach was as novel as his result remarkable. He ruminated on what very basic qualities a reasonable voting method (or in Arrow's terminology a "social welfare function") should possess and proved the only method that satisfied all the criteria is a dictatorship.

1.2 Spatial Models of Elections

While Condorcet demonstrated the intransitivity of majority social preferences and Arrow showed that any voting system was subject to paradox, their results give little indication of the prevalence or importance of the paradoxes for practical purposes. Conceptualizing elections in issue spaces can help in understanding what is likely to happen as opposed to merely what can possibly happen.

A famous result based on spatial model analysis of voting is the median voter theorem [Bla48]. It states that if voters and alternatives exist in a one dimensional issue space (e.g. a spectrum from left-liberal to right-conservative) then when the society chooses between two options represented at points on the spectrum the option that wins is the option that the median voter chooses. Moreover, Black observed that if an election can be represented this way in one dimension then the restriction on voter preferences is such that Arrow's criteria are satisfied. [Bla58] These results do not hold, generally, for elections whose issue spaces are more then one-dimensional. [Bor84] [LBW02].

This paper will use a spatial model to simulate elections under a variety of different conditions to compare the results of different voting methods. Several defintions are nessecary:

- This paper will model elections that occur in two-dimensional **issue spaces**. That is voter and candidate idealogies will be represented by point in two dimensional space. An example of such a space would be a refinement of the liberal-conservative continuum into two continuums for economic and social ideologies.

- Voters are represented by their **ideal points** which are coordinates in the issue space which maximize a voter's utility. Candidate's ideological positions are similarly represented by coordinates in the issue space. A Voter's utility for an alternative will decline as a function of the distance from the candidate to the voter's ideal point.

- A Voter's **preference** (complete transitive ordering of the alternatives) is constructed from the relative distances from the voter to each of the alternatives.

- The distance from a voter to candidate is defined by a **distance metric**.

- A **profile** is the set of all voter preferences. It is constructed from the voters and candidates in the election and the specified distance metric.
- A **voting method** is a social choice function that maps a *profile* to a winning alternative or social choice.
- An **election** consists of a *profile* and a *voting method* that maps the profile to a social choice.

Chapter 2

Visualization Method

2.1 Visualization Creation

In order to understand the behavior of different voting methods in a spatial model I use the visualization method created by Ka-Ping Yee (UC Berkeley). [Yee08] *Visualizations* are colored representations of the results of a series of similar elections that occur in an R^2 issue space like those above. A section of R^2 is divided into a grid. A grid that is 100x100, or 10,000 elections, gives sufficient resolution. For a given visualization:

– A set of *candidates* is generated, generally consisting of between three (most voting methods become equivalent and trivial with two alternatives) and eight candidates(with too many candidates the images become to crowded to be useful), each candidate is assigned a color and idealogical coordinates.

– A set of *voters* and their coordinates is generated from a particular distribution. *To produce informative visualizations, the distributions and scales are chosen so that the area containing all the voters is roughly the size the grid*

– An origin or *balance point* from the mean x coordinate and y coordinate is calculated. The x coordinate of all the voters is summed and then divided by the number of voters, and the same is done for the y coordinate.

– The voters' ideal points are represented as vectors from the balance point and thereby held in constant positions relative to each other.

- Additional parameters are set:
  * A distance norm for the model is chosen.
  * A voting method is selected.
- The voters' balance point is moved to a location in the grid(e.g. the upper-left-hand corner square) while keeping candidate positions fixed. *This means each voter has been moved, by the same amount and in the same direction.*
- A new profile is generated as each voter determines its preferences given their new coordinates, the coordinates of each of the candidates and the distance metric. An election is held and the winner under the proscribed voting method is determined.
- The pixel in the upper-left-hand corner is painted the color associated with the winning candidate
- The balance point is moved to another location(e.g. one spot to the right on the grid). *The voters maintain their positions relative to each other and the candidates who remain static throughout stay in the same positions relative to each other, but position of each voter changes relative to the candidates.*
- Distances and preferences are recalculated, a winner is determined, and the corresponding pixel colored appropriately
- The process is repeated for all locations in the grid.

For a given visualization almost all parameters are held constant. The number of candidates and their positions are exogenously determined and fixed throughout. [1] The n-tuple of voter vectors from the balance point is constant. Only the mass of voters is systematicly moved around the grid.

In figure 2.1 the circle in the upper left is centered on the green candidate's idealogical coordinates, the circle near the bottom represents the red candidate. There are 50 voters, the scattered dots represent the location of each of the voter's ideal points when the balance point is placed in the center of the grid (fig. 2.1).

[1] Considering fixed rather than dynamic candidates could be said to limit the relevance of this method to the study of committees(for which it is reasonable to expect alternatives to have fixed characteristics) rather than elections (for which it is reasonable to expect candidates may project themselves as representing a certain ideology that will maximize their election chances given other candidates

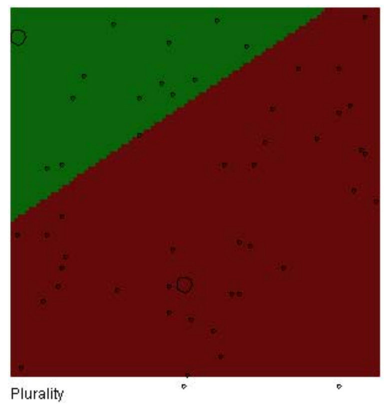


Figure 2.1: 50 Voters; 2 Candidates

The way to interpret the visualization is to focus on a pixel and to imagine that each of the voters(black dots) is shifted by the same amount as the distance from the center of the grid to the focus pixel. An election held with the resulting set of voter ideal points selects the candidate which matches the color of that pixel.

This election in fig. 2.1 involves only 2 candidates and therefore is uninteresting. One thing that should be observed is the appealing shapes of the green and red victory areas. A visual inspection shows that the line dividing the areas is approximately the bisection of the two candidates, a reasonable result which matches expectations that the winning alternative is closer to the the balance point.

2.2 Voting Methods

By generating a set of visualizations simultaneously generated from the same set of profiles it is possible to literally get a picture of how different voting methods behave under the same conditions. There are several voting methods used in this paper:

- **Plurality:** Each voter may vote for one candidate. The candidate with the most votes wins the election.
- **Borda:** Each voter ranks the candidates in preference order. The top ranked candidate gets $n - 1$ where n is the number of candidates in the election. The 2nd ranked candidate receives

$n-2$, the last 0 points. The candidate with the most points wins.

- **Instant Run-off Voting (IRV)**: Each voter ranks the candidates in preference order. Each ballot is assigned to its highest-ranked candidate, and if one candidate has more than half the ballots, that candidate wins. Otherwise, the candidate with the least first-ranked votes is eliminated, and the ballots ranking that candidate highest are reassigned to the next-highest non-eliminated candidate. The counting and elimination process is repeated until there is candidate that receives majority votes.
- **Condorcet:** Each candidate is compared pair-wise to each other candidate. In each matching the candidate that is preferred by a majority of voters beats the other candidate. A Candidate that is unbeaten against all other candidates is a Condorcet Winner. *There is not always a Condorcet Winner, there have been many Condorcet Completion methods suggested, however, rather than using a completion method, a pixel will be left white if there is no winner in the corresponding election.*
- **Approval:** Each voter may vote for as many candidates as desired. The candidate with the most votes wins.

If voters are assumed to be sincere [2] then translating their distances to each candidate into ballots tends to be straight forward:

- **Plurality:** Each voter votes for the least distant candidate.

- **Borda, Condorcet, IRV:** Voters return a preference ordering from least to most distant. These orderings are the ballots that are used for tabulating the results.

- **Approval:** Two methods are used:

  - **log-normal:** In keeping with Ka-Ping Yee's method each voter, in addition to coordinates, is assigned an *approval distance* generated randomly from a log-normal distribution. A voter approves of all candidates within the circle centered on the voter with approving. (ApprovalLN)
  - **mean:** An alternative model is for each voter to compute the mean distance to each of the candidates and vote for all candidates less distant than the mean. (ApprovalM) [MIT91]

[2] Obviously a major assumption

2.3 Additional Parameters

Many analyses of voting methods assume uniform distributions of voters which allows inferences to be made from the geometry of the candidates without having to represent voters explicitly. [Tul67] Simulation makes it unneseccary to restrict distributions to uniform. Yee generates coordinates as two pseudo-random normal deviates. [Yee08] It is also possible to generate more exotic distributions of voters, in particular Bi-Modal normal, distributions. In the following section I assign voter coordinates using pairs of values from a pseudo-random uniform real number generator.

Chapter 3

Visualizations

3.1 Large Distributions

Yee simulates elections with 200,000 voters. [Yee08] As a result his winning areas have smooth, if sometimes erratic, borders. These large simulations are useful in understanding what the expected value (or winner) of an election is for a given set of parameters. However, it gives little indication of the variability of the results. The largest distributions I report are for 1000 voters, enough at least for uniform distributions to approximate Yee's results.

The first visualization, represents an election with 1000 voters and 3 candidates. (fig 3.1) There are 6 images, the top-left image is not a visualization, it is a representation of the election: the three candidates and the 1000 voters centered on the center of the grid. The same candidates and voter distribution is used across all 6 images. In the top row the results for plurality and Borda methods are also reported. In the bottom row are results for IRV, Condorcet and Approval(with log-normal generated approval distances). With this relatively large and uniform distribution of voters and three fairly evenly spaced candidates the results are similar across methods. Plurality, Borda and Condorcet look nearly identical. The shape of the red area is curved for IRV and the borders in ApprovalLN visualization are uneven foreshadowing an instability that seems to be endemic to the method in this model.

Figure 3.2 is generated from another uniform distribution of 1000 voters and a new arrangement of three candidates. Yee calls the effect on green's winning area under plurality "squeezing out"; it is a well

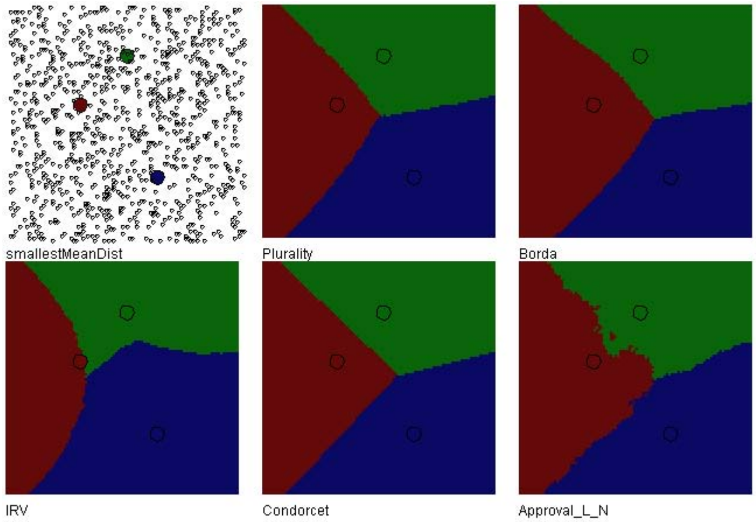


Figure 3.1: 1000 Voters; 3 Candidates

known property of the plurality and IRV methods that they tend to exclude centrist candidates.

In figure 3.2 Borda and Condorcet behave in an appealing fashion the borders of the green area neatly cut the space between the green candidate and red and blue candidates. It is slightly harder to see that the green region in Borda cuts a wider swath then under the Condorcet method. The Borda method frequently produces this result where the interior candidates winning region bulges. Again, conversely, plurality and IRV tend to favor alternatives that are on the outside of of the *candidate* distribution Borda tends to favor candidates that are at the interior of the candidate distribution.[1] ApprovalLN continues to behave somewhat erraticly, splotches of blue appear inside the green region, with one blue patch right next to the green candidate's position.

A reaction to the aesthetics of this image motivates the definition of several other characteristics of voting methods in this model. Yee

[1]It is important to emphasize that this effect depends on a candidate's position relative to other candidates as much as voters. It is a demonstration of the Borda method's failure to satisfy the Independence of Irrelevant Alternatives Criteria(IIA). And implies the common criticism of Borda - it is highly and predictably susceptible to strategic nominations. IIA says that if a social welfare function ranks candidate A over candidate B then the introduction or subtraction of another candidate should not affect the societal preference of A over B

observed that the winning regions were sometimes disjoint as is the blue regions with approvalLN. *Monotonicity of the Visualization* is the expectation that if Candidate A wins for an election with the balance point at a given spot, and then the balance point is moved towards that Candidate A and away from all other candidates that candidate should continue to win.(Violation of this can be seen moving straight up from the blue candidate, green wins but a little higher up still blue wins.)

Another normative quality is the expectation that a candidate should be contained in its winning regions as opposed to the results for the green candidate under Plurality and IRV in this simulation.

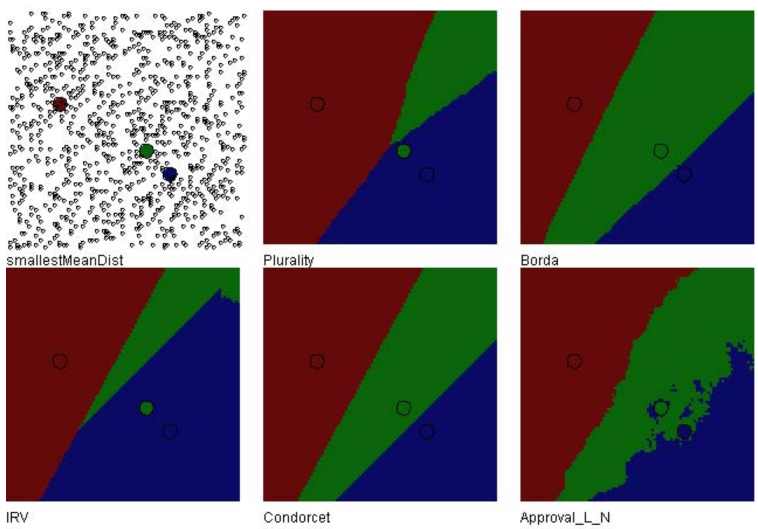


Figure 3.2: 1000 voters; 3 candidates

3.2 Decreasing Populations

Figure 3.3 is generated with 1000 uniformly distributed voters and 5 candidates. As, before Condorcet and Borda behave similarly and smoothly. Plurality is almost as good with just a little squeezing out in evidence. IRV and approvalLN continue to behave the most erraticly.

Figure 3.4 shows the results with the same 5 candidates as in figure 3.3 and uniform populations of decreasing sizes. In addition to the

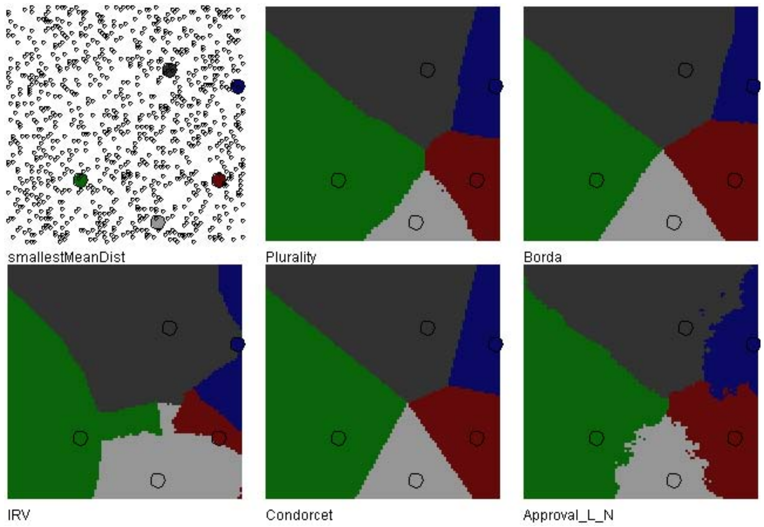


Figure 3.3: 1000 voters; 5 candidates

voting method included in the previous figures the approval method using the mean distance voter model is shown. The top row has a population of 250 voters, the second row has 100 voters and the images in the third and fourth rows are results of elections with 35 and 15 voters respetively. These are diffrent elections with different sizes of voters generated from the same distribution. Elections of this size are important and far more common than larger elections. These sizes are representative of committees and elections that make decisions based on aggregate opinion.

For presentation purposes the voters are represented on only the plurality image for each set of images. As expected, the winning regions tend to degenerate going from the top two rows to the bottom two rows. A white region emerges in Condorcet and grows as the size of the population declines. This means that there are more cases in which there is no Condorcet winner because the majority preference is cyclical. Both approval models lose any any significant coherence. Plurality remains remarkably consistent, and probably by chance seems to actually recover when going from 35 voters to 15.

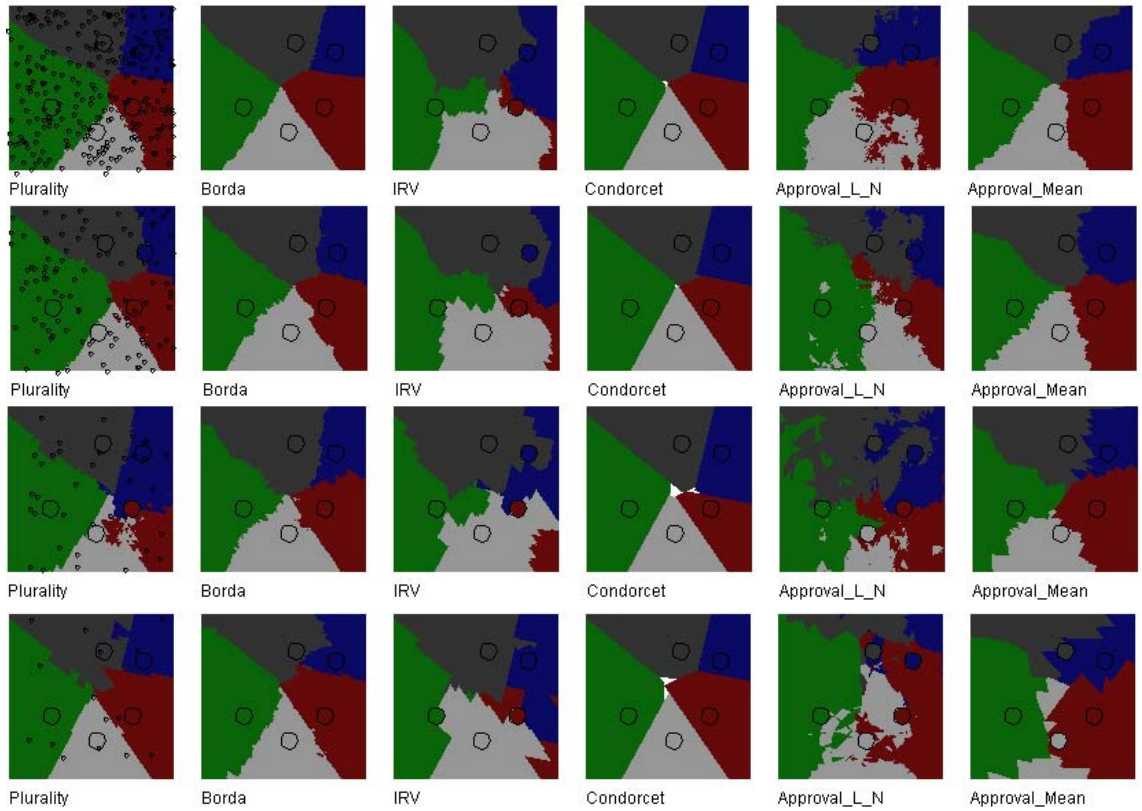


Figure 3.4: 250, 100, 35, 15 Voters; 5 candidates

3.3 Alternative Norms

For the previous visualizations the distance norm, as in Yee has been Euclidean. While Euclidean distances are frequently used there is little reason to suppose they are the most accurate depictions of voter perceptions. For example it seems much more plausible that voters use a Manhattan or taxicab norm when evaluating the size of the mismatch between their ideal and a given candidate. In other words to the extent a spatial model is applicable it seems more natural for a voter to consider the differences for each issue and add them up rather than to square the difference in each dimension and then take the square root of the sum.

In addition to alternative norms there are other spatial models of voter prefence which can be simuluated in this framework. Traditionaly spatial-models have been proximty based, that is voter preference declines as a monotonic function of distance. However, there is empirical evidence that a directional model of voter preferences or a hybrid of directional and proximity best models voter preferences. [MLR91] [CE03] Results are not reported here.

All the images in figure 3.5 use the same three candidates and the same distribution of 35 voters. The top eight images use Euclidean distances, and bottom eight use the taxicab norm. Each set, in addition to the six images in previous figures, includes two additional images. In the top left is a visualization representing a potential normative criteria. The normative criteria used does not rely on the ordinal rankings of the voting methods that have been analyzed. Rather it uses the actual distances for a voter to each candidate and determines which candidate minimizes the *smallest mean distance(SMD)*.[2].

The image in the lower-right hand corner is an alternative baseline. In the *random dictator method* for each election a voter is randomly selected, that voters preferred candidate wins. This is a sort of lower bound on election method peformance.

Notice how SMD changes very little across the two models but Borda and Condorcet change their shapes drasticly. Under Euclidean distances Borda and Condorcet look a lot like the normative criteria but under the Manhattan norm they look very different than the normative image.

3.4 Normative Overlay

One strategy for making comparisons easier is to overlay the normative criteria on the other images. In figure 3.6, as done previously, each pixel is colored to match the color of the winner, but a black stripe is added when the winner at that point matches the winner at the corresponding for the normative criteria. For example the left half of the plurality image is colored green and the left portion of the green area is striped to show that the green is also the SMD winner but in

[2] If voters are honest this is equivalent to the range-voting method advocated by Warren Smith [Smi00]

the unstriped region the SMD winner is blue. The plurality image is particularly interesting in this set because of the stripes - definitely not monotonic. Also, in ApprovalM the blue region is massive, well over half of the grid, while the normative region is broken neatly into thirds. It is not uncommon for one candidate in ApprovalM to have an inordinately large region.

3.5 Large Numbers of Candidates

Up until now the elections that have been simulated have included a relatively small number of candidates. In this section elections with 6 and 8 candidates are simulated as the number of candidates increases the performance of the voting methods decreases dramticly and becomes more erratic.

In figure 3.7 this chaos of many-candidate elections is well demonstrated. There are three sets of elections. Each election has a population of 35 voters generated from uniform distributions. The first election involves 4 candidates, the second election 6 candidates, and the third elections decides between 8 candidates.

Notice how under ApprovalM with 6 Candidates green dominates almost the entire region with 8 candidates gray does. With 8 candidates ApprovalLN starts to look like Random Dictator as much as anything else. As the number of candidates increases the Condorcet Paradox regions grow in size and number. None of the centrist candidates ever wins under plurality with 8 candidates. IRV does not appear to perform any better than plurality. Borda performs the most consistently.

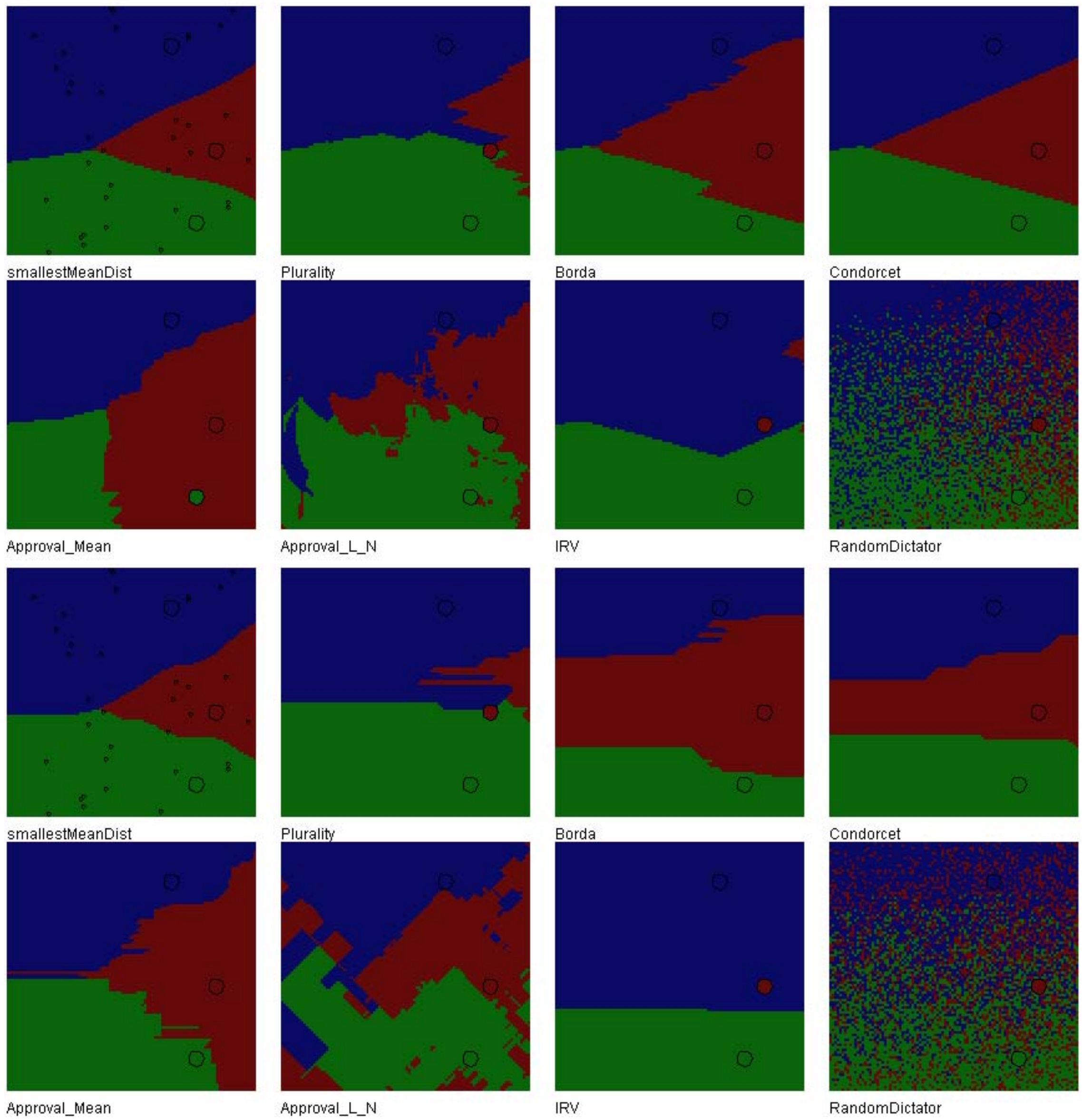


Figure 3.5: 35 voters; 3 candidates - Euclidean vs Sum of Absolute Distances

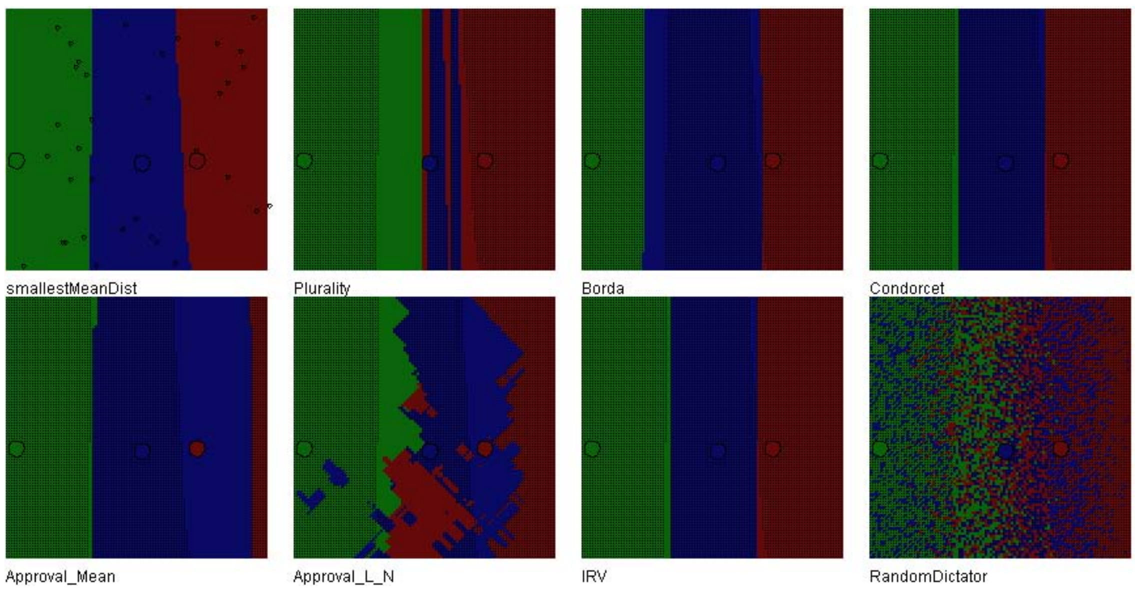
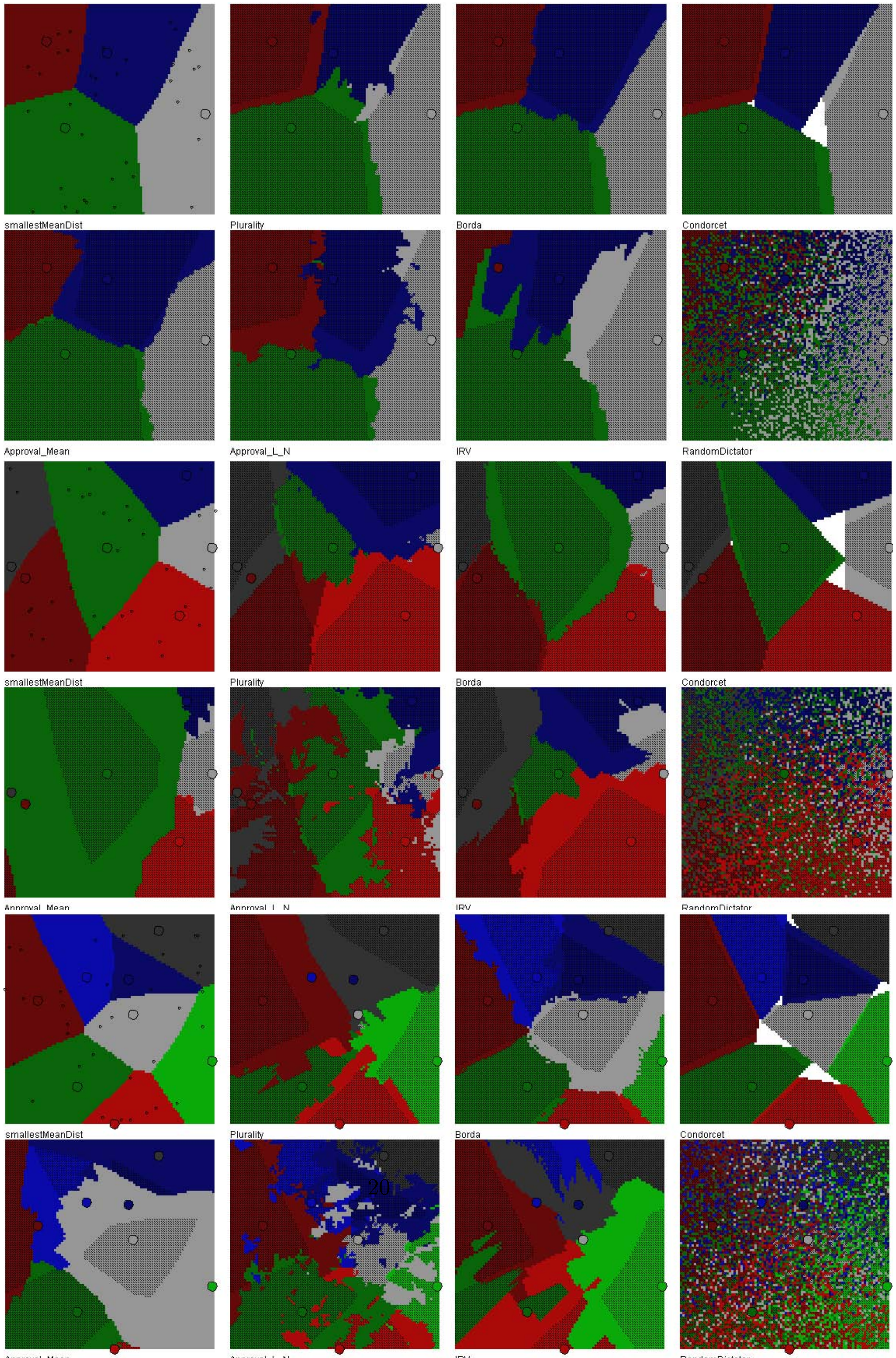


Figure 3.6: 35 Voters, 3 Candidates - Normative Overlay


smallestMeanDist
Plurality
Borda
Condorcet
Approval_Mean
Approval_L_N
IRV
RandomDictator
smallestMeanDist
Plurality
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Approval_Mean
Approval_L_N
IRV
RandomDictator
smallestMeanDist
Plurality
Borda
Condorcet
Approval_Mean
Approval_L_N
IRV
RandomDictator

Chapter 4

Repeated Trials

4.1 Repetition

In this section I modify the visualization technique to understand the variable behavior of the voting methods. While the isolated simulations of elections in the previos chapter provide anecdotal evidence on the behavior of different voting methods repetition makes more useful comparisons possible.

The concept of repeated trials is fairly straightforward in the visualization framework. The follwing images are generated by aggregating the results of a 1000 visualization generated from the same parameters. That is the candidates remain fixed in all the trials. In each trial, the set, but not number, of voters changes but is derived from the same distribution.

Over the course of the simulation, the number of times a particular candidate wins at a particular spot in the grid is recorded. The color and shade of the corresponding pixel is dependent on which candidate wins and how frequently. It is therefore possible to see from the image which candidate usually wins at partiular spot but also how consistent that result is.

The following figures display the results of a series of elections with four candidates. Three of the candidates are at the vertices of an equilateral triangle and the fourth is at the triangle's centroid. The elections simulated are small, the voting population consists of nine ideal points selected from a uniform distribution. There are 1000 trials.

In each trial a set of nine voters is generated and the election is held at all 10,000 points in the grid. Then a new set of nine voters is created and another 10,000 elections are held. The results of the 1000 trials are aggregated in figure 4.1.

The white areas represent points on the grid where no one candidate was the winner a majority of times. That is at each pixel there were 1000 elections held and no canididate won more than 500 of them. In the random dictator image it is easy to see that moving from the center closer to an outer candidate, green for example, the pixels become light-green and then steadily darker as the proportion of green victories approaches one.

In figure 4.1 it is easy to see pluarality and Borda exhibiting their usual behavior. Under plurality the central candidate, red, is shorted while under Borda the red candidate's region bulges. ApprovalM is completely red, this is unsuprising, since in most cases the distance from a voter to the red candidate is less then the mean distance.

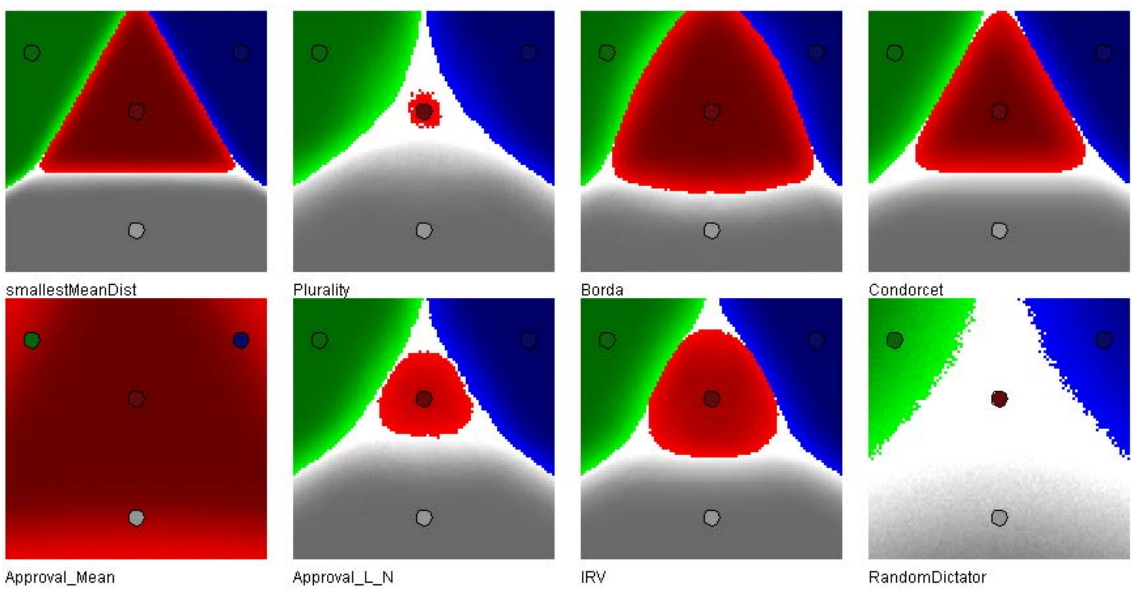

Figure 4.1: 9 Voters; 4 Candidates; 1000 Trials, Equilateral Triangle

Figure 4.2 is generated by the same simulation as figure 4.1 except that rather than populations of 9 voters, visualizations are generated for

1000 simulations with 33(top) and 129(bottom) voters. In general the regions appear more precisely defined. The transition from one dark shade to another happens over a smaller area. IRV with 129 voters looks very different than IRV with 9 voters. It is also interesting to note that the small patch of red that exists for plurality with 9 voters disapears when elections contain more voters. This is a surprising result result of a voting method apparently returning better, more consistent, results for fewer voters.

4.2 Consistency

It is possible to quantify the effect on consistency that occurs within each voting method when the number of voters increases. The same simulation seen in figures 4.1 and 4.2 is run for 9, 17, 33, 65, and 129 voters. (figure 4.3) For each pixel the number of times that the candidate who most frequently wins the elections centered on that pixel is recorded. The average proportion of elections won by the winningest candidate in each pixel over the 10,000 pixels in that image gives a an index of consistency. That is a image with a high consistency index is one where the winner of an election is likely to remain constant when the election simulation is repeated with the same parameters.

It is unsurprising that the consistency level improves for all voting methods as the number of votes incresing. The progress is not homogeneous, plurality starts out less consistent then Borda but steadily over takes it by the time elections are larger. This is most likely an idiosyncracy of the contrived geometry of this particular simulated election.

Additionally, it is important not to over interpret the index since ApprovalM has the highest rating but is consistent only at picking the red central candidate even when the balance point is far away outside of the triangle. One more interesting phenomenon is that the consistency or the random dictator slowly but consistenly drops.

4.3 More Images

Figure 4.4 is a side by side comparison of the same parameters using euclidean and taxicab distances. The most signficant thing that this comparison illustrates is the model dependency of the simulation. This is particularly evident in the ApprovalM visualization.

Figure 4.5 simulates elections with 4 candidates located at the vertices of a square. Notice how the white space in the middle varies between methods as well as the shading around the 4 border areas; Borda is curved, while Condorcet is straight.

Figures 4.6 and 4.7 are repeated trial simulations of elections with 6 and 7 candidates respectively.

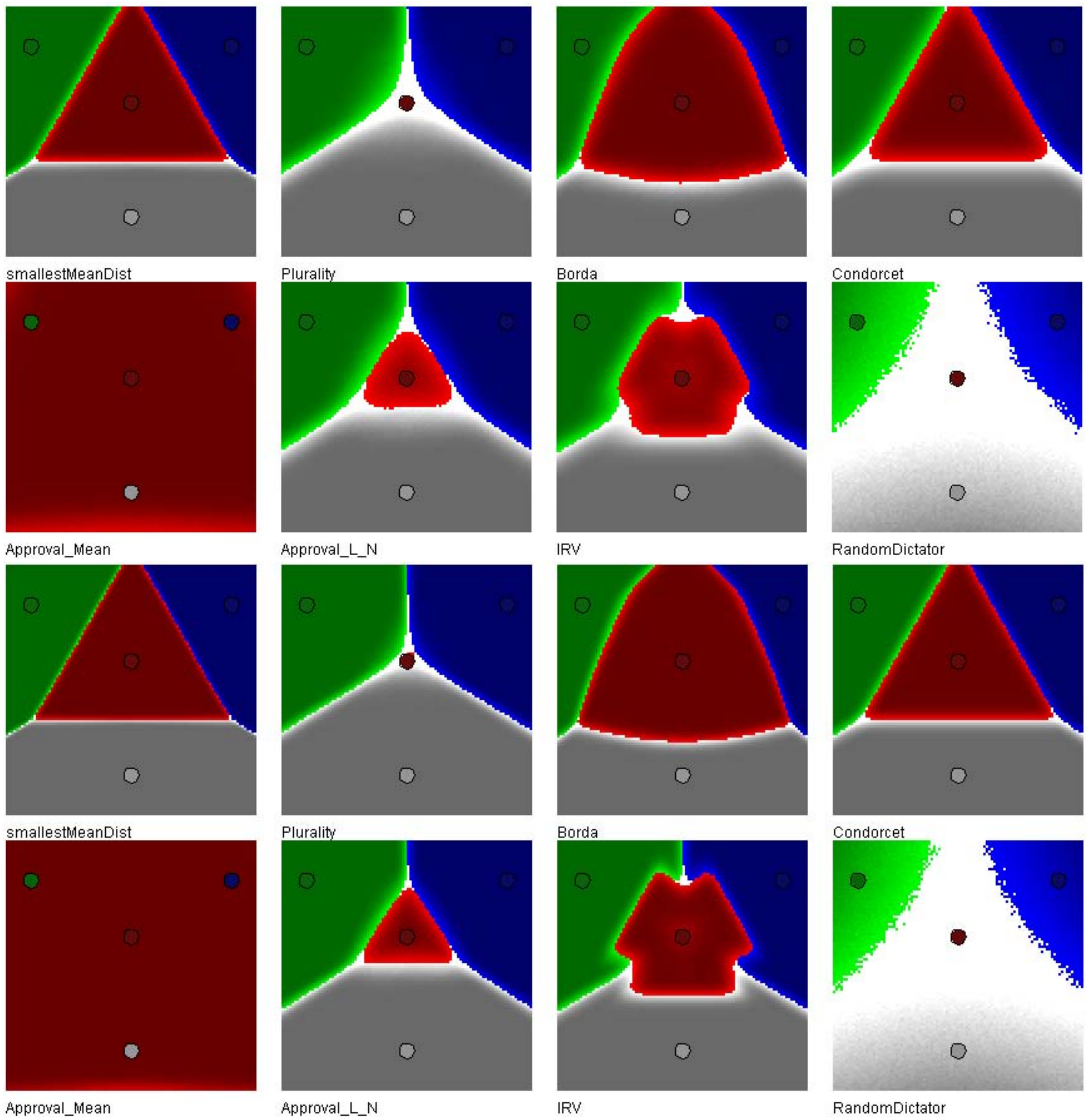

Figure 4.2: 33 Voters, 129 Voters; 4 Candidates; 1000 Trials, Equilateral Triangle

Method	9-Voters	17-Voters	33-voters	65-Voters	129-voters
SMD	0.930	0.946	0.961	0.971	0.980
Plurality	0.838	0.883	0.917	0.944	0.962
Borda	0.856	0.888	0.920	0.941	0.958
Condorcet	0.854	0.884	0.913	0.937	0.955
Approv_Mean	0.915	0.955	0.976	0.987	0.993
Approv_L_N	0.837	0.873	0.905	0.928	0.947
IRV	0.827	0.853	0.887	0.915	0.939
Rand. Dict.	0.581	0.574	0.571	0.569	0.568

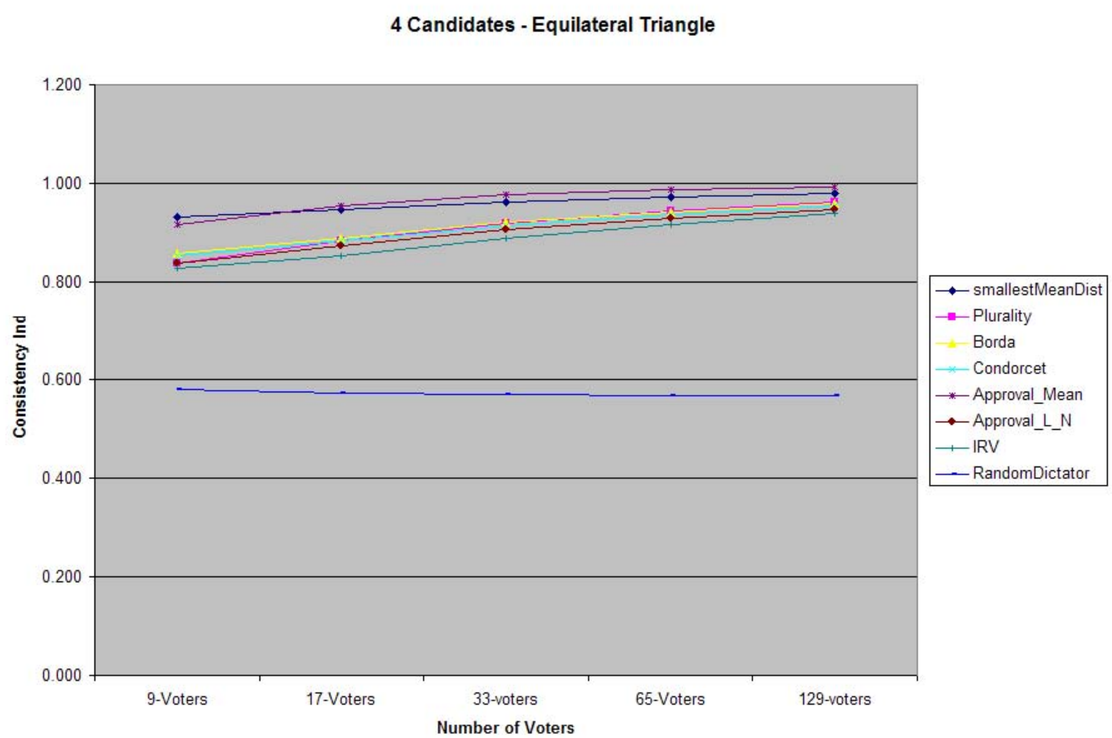

Figure 4.3: Consistency Index; 4 Candidates; 9,17,33,65,129V; 1000 Trials, Equilateral Triangle

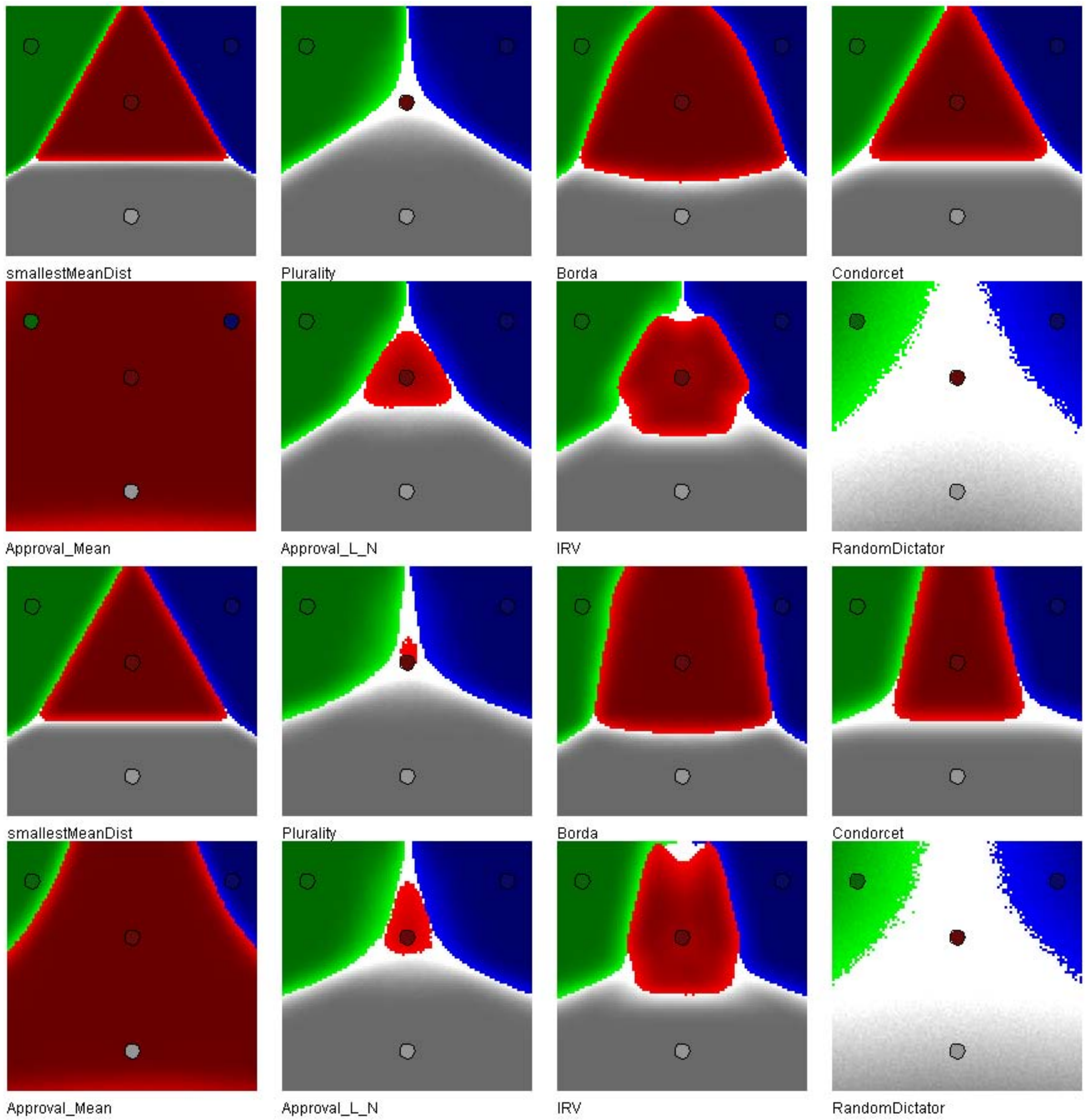

Figure 4.4: Euclid and SumAbs; 4 Candidates; 33V; 1000 Trials, Equilateral Triangle

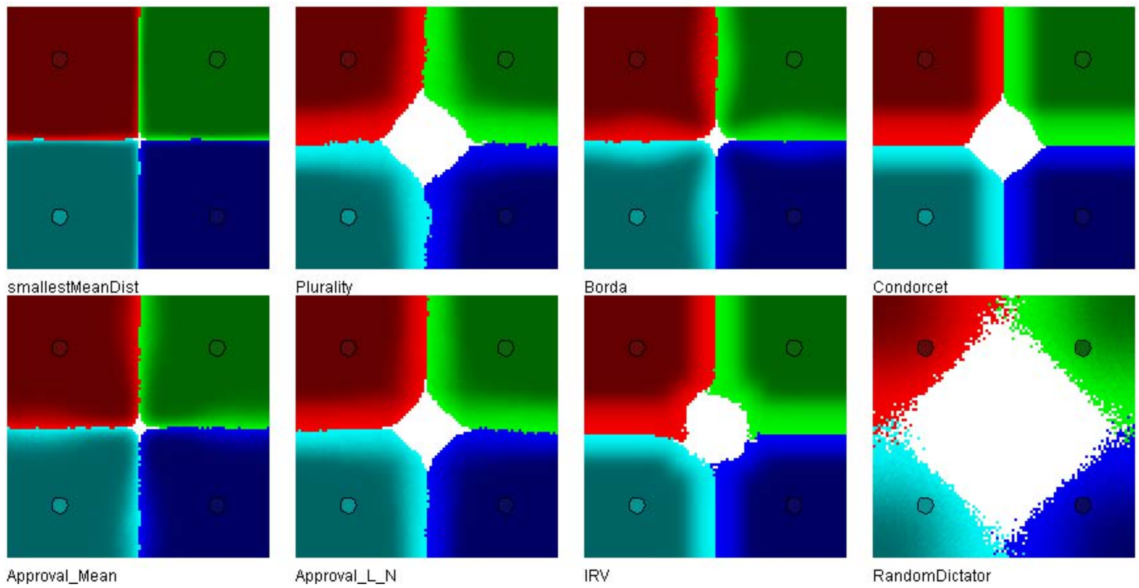

Figure 4.5: Euclid; 4 Candidates; 33V; 1000 Trials, Square

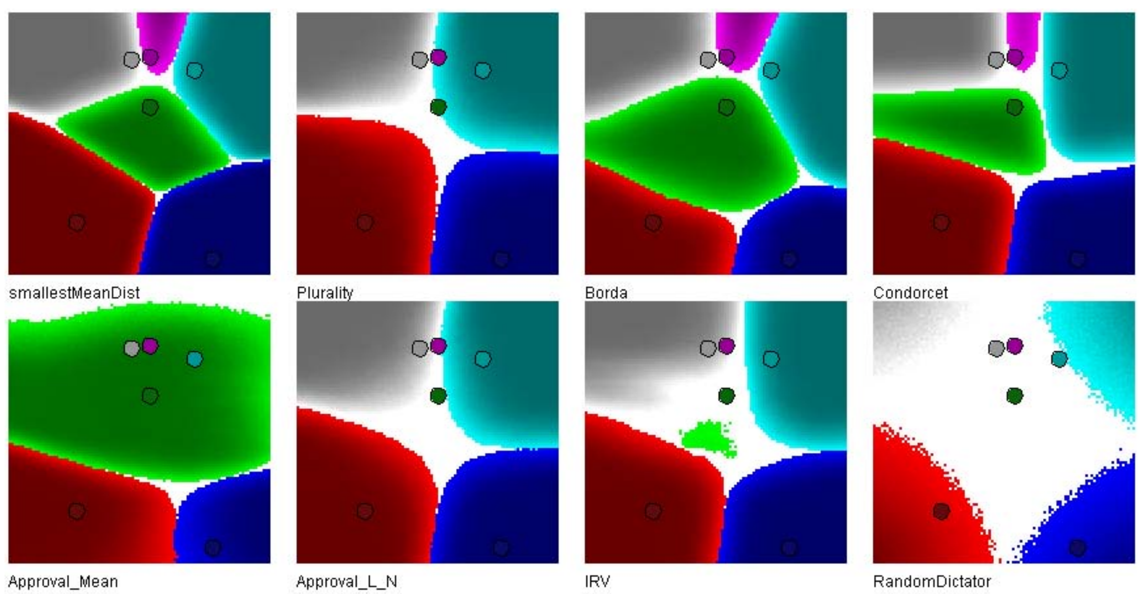

Figure 4.6: sumAbs; 6 Candidates; 17V; 1000 Trials

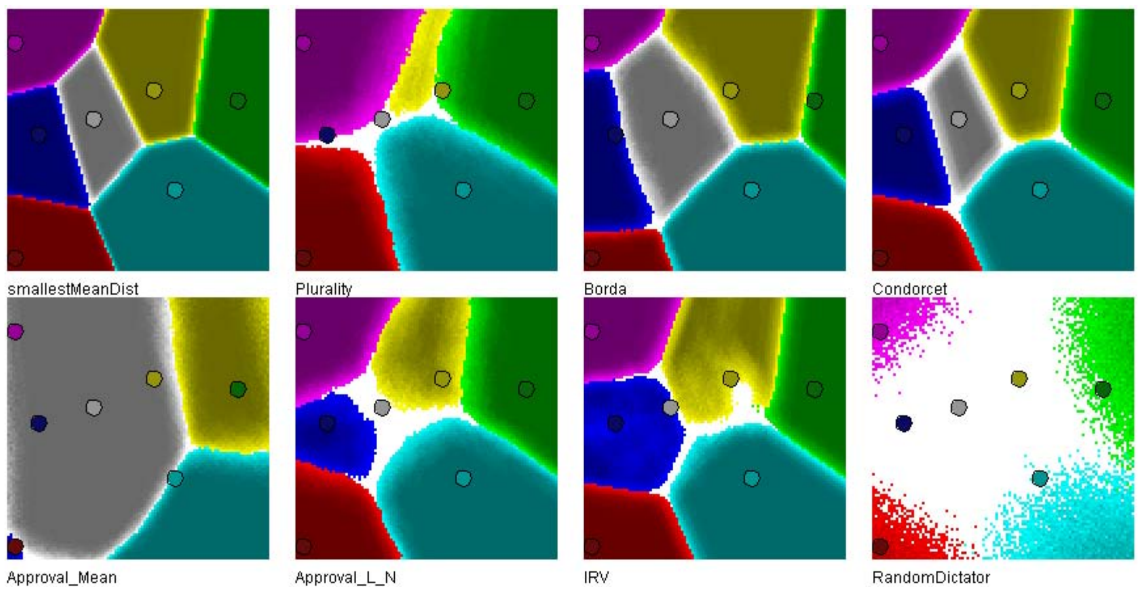

Figure 4.7: Euclid; 7 Candidates; 65V; 1000 Trials

Chapter 5

Conclusion

Voting paradoxes have been a subject of academic debate since at least the 18th century. Even Arrow's result, that no voting method could satisify a small "reasonable" set of criteria, almost paradoxically, served only to reinvigorate the field. Subsequent analysis have tested the limits of the critera, revealing even stronger impossibility results. While Arrow's theorem is often interpreted to say that there is no social welfare function. It is of substantial practical importance to understand the effects of an aggregation method.

The visualizations in this paper compare some of the more commonly known voting methods. There are many, more recently proposed, methods that can be expected to perform in a more desirable manner but they may too unintutive to be adopted for social or political decisions and again will never meet Arrow's standard. The significance of the visualizations is to demonstrate the variation in the results that different voting methods produce. By looking at a visualization for a few seconds it is possible to gain an intutitve grasp of the differences between methods by a systematic comparison of 10,000, or the case of the repeated trials simulations the results of ten million, elections. The robustness of each method in the context of peer methods, imposed normative criteria or even to itelf can be observed.

The visualization method can be used with more sophisticated models of elections. Strategic voters can be modeled for each election to find out whether the methods that tend to perform better with sincere voters still outperform the other methods. Elections with the same set of voter and candidate locations and election method but different voter strategies can be directly compared, with the normative overlay method, to generate visual contrasts of the differences. The visualization method still has insights to offer.

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