Math 151, Probability
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1 Markov Chains

Markov Chains

Markov chain is a mathematical system that undergoes transitions from one state to another which depends only on the current state and is independent of the states preceding it.

Example. Consider a two-state weather forecast model, where weather is classified as sunny or rainy.

- If today's forecast is rainy, then there is a 50% chance that it will be rainy the following day and a 50% chance that it will be sunny the following day.
- But if today's forecast is sunny, there is a 5% chance that it will be rainy the next day and 95% chance that it will be sunny the following day.

Suppose today's forecast is sunny. Then the probability that it will be sunny in two days from now is:

$$\begin{aligned} P(\text{sun 2 days}|\text{sun now}) &= P(\text{sun in 2 days}|\text{sun tom})P(\text{sun tom}|\text{sun now}) \\ &\quad +P(\text{sun 2 days}|\text{rain tom})P(\text{rain tom}|\text{sun now}) \\ &= (.95)(.95) + (.5)(.05) = 0.9275 \end{aligned}$$

While the probability that it will be rainy two days from now is:

$$\begin{aligned} P(\text{rain 2 days}|\text{sun now}) &= P(\text{rain in 2 days}|\text{sun tom})P(\text{sun tom}|\text{sun now}) \\ &\quad +P(\text{rain 2 days}|\text{rain tom})P(\text{rain tom}|\text{sun now}) \\ &= (.05)(.95) + (.5)(.05) = 0.0725. \end{aligned}$$

Continue this iteration and after $n >> 0$ steps, we reach what is known as invariant (or equivariant or stationary or steady state or equilibrium) distribution.

Stochastic Processes

- Suppose we are observing a system with a finite number of possible states, whose state changes in an unpredictable (*stochastic, random, probabilistic*) fashion from one time step to the next.

  This is called a **discrete stochastic process**.

  Examples:

- **the number of phone lines in use in an office at the beginning of each minute,**
- **the state of the weather (rainy or fair) in Claremont at 9AM each morning,**
- **the price of a particular stock at closing each day.**

Notation

We write the process as $X_1, X_2, \ldots X_n, \ldots$, where X_n is the observation at the nth time step.

X_1 is the initial state.

Since the process is stochastic, it is described by the probabilities:

$$P\{X_{n+1} = x_{n+1} | X_1 = x_1, X_2 = x_2, \ldots X_n = x_n\}$$

Definition: A Markov Chain is a discrete stochastic process where the probabilities of each state *only depend on the current state*, not the previous ones:

$$P\{X_{n+1} = x_{n+1} | X_1 = x_1, X_2 = x_2, \ldots X_n = x_n\} = P\{X_{n+1} = x_{n+1} | X_n = x_n\}$$

and the multiplication rule gives:

$$P\{X_1 = x_1, X_2 = x_2, \ldots X_n = x_n\} = \underline{\hspace{8cm}}$$

Same convention we've been using: capital letters are random values, or random "variables", while lower case letters are the values that they take on.

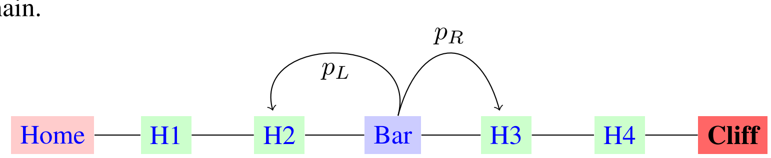
Markov processes are sometimes called "memoryless".

$P\{X_1 = x_1, X_2 = x_2, \ldots X_n = x_n\} = P\{X_1 = x_1\} P\{X_2 = x_2 | X_1 = x_1\} \ldots P\{X_n = x_n | X_{n-1} = x_{n-1}\}$

1.1 Example: A Random Walk

A Random Walk

A fellow leaves the bar and staggers along the street. Suppose the probability of moving left at each step is $p_L = 1/3$, and the probability of moving right at each step is $p_R = 2/3$. Then his position at time n is a Markov Chain.



What are the defining probabilities for this Markov Chain?

$P(X_n = k | X_{n-1} = k-1) = 2/3, P(X_n = k | X_{n-1} = k+1) = 1/3$, and the other probabilities are zero.

1.2 The Transition Matrix

The transition matrix

The probabilities of going from one state to another are transition probabilities. If they don't depend on time (i.e. on n), then the process is called a finite (state) Markov Chain with stationary probabilities. We can organize the transition probabilities in a Transition Matrix, $P = (p_{ij})$, where

$$Prob\{X_{n+1} = s_j | X_n = s_i\} = p_{ij}$$

Ex. 3.10.5 Occupied phone lines

$$P = \begin{array}{c|cccccc|} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ \hline b_0 & .1 & .4 & .2 & .1 & .1 & .1 \\ b_1 & .2 & .3 & .2 & .1 & .1 & .1 \\ b_2 & .1 & .2 & .3 & .2 & .1 & .1 \\ b_3 & .1 & .1 & .2 & .3 & .2 & .1 \\ b_4 & .1 & .1 & .1 & .2 & .3 & .2 \\ b_5 & .1 & .1 & .1 & .1 & .4 & .2 \end{array}$$

The state b_i represents i telephone lines in use.

If all five lines are currently in use, what is the probability that exactly 4 will be in use at the next time step? If no lines are currently in use, what is the probability that at least one will be in use at the next time step?

Note that the entries of each *row* sum to one. A matrix with this property is called a *stochastic matrix*.

Show that the entries of P^2 give the two-step transition probabilities.

Transitioning

Show that the entries of P^2 give the two-step transition probabilities.

Let's say we want to find the probability of starting in state 3 and going to state 2 in TWO steps. We can use the $(3, 2)$ element of the matrix P^2. Why?

In order to get from 3 to 2 in two steps, we have to start at 3, go anywhere, and then move to step 2. For example, we could start at 3: (move to 1 AND move to 2) OR (move to 4 AND move to 2).

$$\begin{aligned} P(X_{n+1} = 1, X_{n+2} = 2 | X_n = 3) &= P(X_{n+1} = 1 | X_n = 3) P(X_{n+2} = 2 | X_{n+1} = 1) \\ &= p_{31} p_{12} \end{aligned}$$

Because we can use any path to get from 3 to 2 ("OR"), we add the probabilities,

$$P(\text{going from 3 to 2 in two steps}) = \sum_i p_{3i} p_{i2}$$

and $\sum_i p_{3i} p_{i2}$ is the (3,2) entry of P^2.

P^2

Ex. 3.10.5 Occupied phone lines

$$P^2 = \begin{pmatrix} .1 & .4 & .2 & .1 & .1 & .1 \\ .2 & .3 & .2 & .1 & .1 & .1 \\ .1 & .2 & .3 & .2 & .1 & .1 \\ .1 & .1 & .2 & .3 & .2 & .1 \\ .1 & .1 & .1 & .2 & .3 & .2 \\ .1 & .1 & .1 & .1 & .4 & .2 \end{pmatrix} \times \begin{pmatrix} .1 & .4 & .2 & .1 & .1 & .1 \\ .2 & .3 & .2 & .1 & .1 & .1 \\ .1 & .2 & .3 & .2 & .1 & .1 \\ .1 & .1 & .2 & .3 & .2 & .1 \\ .1 & .1 & .1 & .2 & .3 & .2 \\ .1 & .1 & .1 & .1 & .4 & .2 \end{pmatrix}$$

$$P^2_{(3,2)} = 0.1 \cdot 0.4 + 0.2 \cdot 0.3 + 0.3 \cdot 0.2 + 0.2 \cdot 0.1 + 0.1 \cdot 0.1 + 0.1 \cdot 0.1 = \sum_i p_{3i} p_{i2}$$

Properties of the transition matrix

- The n-step transition probabilities are given by the entries of P^n.
- Given an initial probability distribution vector, v, the product:

$$w = vP$$

  gives the probabilities of all states after one time step.
- Find the invariant distribution for the phone matrix. (Example: 3.10.3)

  Let $v = [.5, .3, .2, 0, 0, 0]$. Repeatedly multiply v by P.
- Under certain conditions (all entries of $P^n > 0$ for some n), *all* initial probability vectors converge to the (unique) invariant distribution. *These Markov Chains are called ergodic* .
- Note that for an ergodic Markov Chain, the unique invariant distribution gives the proportion of time that the process spends, on average, in each state. Intuitively: let P_k be the proportion of time spent in state k. Then the proportion of time spent in state j is

$$P_j = \sum_k P_k P_{kj}$$

Note: P_{kj} is the probability of a transition from state k to state j. But this is the same equation that the invariant distribution must satisfy, so they are indeed the same values (note that the proportions must sum to 1 just as the probabilities must).

1.3 Exercise

Example

A certain person goes for a run each morning. When he leaves his house for his run he is equally likely to go out either the front or the back door; and similarly when he returns he is equally likely to go out either the front or back door. The runner owns 5 pairs of running shoes which he takes off after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running he runs barefooted. We are interested in determining the proportion of time that he runs barefooted. **(a)** Set this up as a Markov chain. Give the states and the transition probabilities. **(b)** Determine the proportion of days that he runs barefooted.

Answer: One way to set up the Markov Chain is:

Let State i be "there are i shoes at the door he leaves from (before he puts the shoes on)", where $i = 0, 1, \ldots 5$. We can ignore *which* door the runner leaves, since he is equally likely to leave from, or return to, each door, so they will have equal probabilities of having a certain number of shoes.

To figure out the transition probabilities, we realize that he is equally likely to return to the same door as he is to return to the other door, so, *after* the run (after he takes his shoes off), there are either going to be i and $5 - i$ shoes at the two doors (scenario A: he returns to the same door he left from), or $i - 1$ and $6 - i$ shoes at the two doors (scenario B: he returns to the door he did *not* leave from), where the two scenarios have equal likelihood. The next morning he leaves from either door with equal probability. This gives the transition probabilities:

$$P(i,i) = P(i,5-i) = P(i,i-1) = P(i,6-i) = \frac{1}{4}.$$

and all other $P(i, j)$ are 0. We need to check the extreme cases. In the case that there are no shoes at the door ($i = 0$), he runs barefoot, so there is no possibility of changing the numbers of shoes at the doors at the end of the run. There is the possibility of choosing the other door for the initiation of the next day's run. Thus:

$$P(0,0) = P(0,5) = \frac{1}{2}.$$

If $i = 5$ then everything works out as in the general case. If $i = 3$ then $i - 1 = 5 - 1$ and $i = 6 - i$, so

$$P(3,3) = P(3,2) = \frac{1}{2}.$$

Putting this together we have the following transition matrix:

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4/ & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \end{bmatrix}$$

To find the proportion of days that he runs barefooted, we find the invariant distribution of the matrix, which is:

$$\Pi = [1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6].$$

Thus, he runs barefooted (the state “0”) 1/6 of the time.