Math 151 – Probability
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Functions of random variables: simulation

The motivation

We can simulate real numbers on the interval [0,1]. We'd like to be able to simulate variables from other distributions. In fact, we'd like to be able to simulate observations from the following distribution:

$$\begin{aligned} \text{pdf: } g(\star) &= \lambda e^{-\star\lambda} \qquad \star \geq 0 \\ \text{cdf: } G(\star) &= 1 - e^{-\star\lambda} \qquad \star \geq 0 \end{aligned}$$

The set up

Let X be a uniform [0,1] random variable. That is, $f_X(x) = 1 \quad 0 \leq x \leq 1$; $F_X(x) = x \quad 0 \leq x \leq 1$.

Let $Y = G^{-1}(X)$. What is the distribution of Y?

Note:

$$\begin{aligned} X &= 1 - e^{-Y\lambda} \\ Y &= -\ln(1 - X)/\lambda \end{aligned}$$

The solution

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(G^{-1}(X) \leq y) \\ &= P(X \leq G(y)) \\ &= F_X(G(y)) \\ &= G(y) \end{aligned}$$

That is, if we let $Y = G^{-1}(X)$, then the random variable Y will have exactly the distribution for which we were hoping.

The implications

The relationship above holds in both directions. That is, if Y has *any* distribution G, then $X = G(Y)$ will have a uniform distribution on [0,1].

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(G(Y) \leq x) = P(Y \leq G^{-1}(x)) \\ &= G(G^{-1}(x)) = x \qquad 0 \leq x \leq 1 \end{aligned}$$

Which proves that X has a uniform distribution on [0,1].

How does it work?

1. (a) Find a random uniform observation, x^*
   (b) $G^{-1}(x^*)$ will be the random exponential observation we simulate.
2. (a) Find a random observation from any distribution, y^*
   (b) $G(y^*)$ will be the random uniform [0,1] observation we simulate