Math 151 – Probability
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Functions of random variables: simulation

The motivation

We can simulate real numbers on the interval [0,1]. We'd like to be able to simulate variables from other distributions. For example, we'd like to be able to simulate observations from the following distribution:

$$\begin{aligned} \text{pdf: } g(\star) &= \lambda e^{-\star\lambda} \qquad \star \geq 0 \\ \text{cdf: } G(\star) &= 1 - e^{-\star\lambda} \qquad \star \geq 0 \end{aligned}$$

The set up

Let X be a uniform [0,1] random variable. That is, $f_X(x) = 1 \quad 0 \leq x \leq 1$; $F_X(x) = x \quad 0 \leq x \leq 1$.

Let $Y = G^{-1}(X)$. What is the distribution of Y?

Note that using the example distribution above:

$$\begin{aligned} X &= 1 - e^{-Y\lambda} \\ Y &= -\ln(1 - X)/\lambda \end{aligned}$$

The solution

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(G^{-1}(X) \leq y) \\ &= P(X \leq G(y)) \\ &= F_X(G(y)) \\ &= G(y) \end{aligned}$$

That is, if we let $Y = G^{-1}(X)$, then the random variable Y will have exactly the distribution for which we were hoping (regardless of the distribution we are trying to simulate).

The implications

The relationship above holds in both directions. That is, if Y has *any* distribution G, then $X = G(Y)$ will have a uniform distribution on [0,1].

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(G(Y) \leq x) = P(Y \leq G^{-1}(x)) \\ &= G(G^{-1}(x)) = x \qquad 0 \leq x \leq 1 \end{aligned}$$

Which proves that X has a uniform distribution on [0,1].

How does it work?

1. (a) Find a random uniform observation, x^*
   (b) $G^{-1}(x^*)$ will be the random, for example, exponential observation we simulate.
2. (a) Find a random observation from any distribution, y^*
   (b) $G(y^*)$ will be the random uniform [0,1] observation we simulate