8.8 Solutions to Exercises

1. In this exercise, $\xi_0 = 0.9$, $\xi_1 = 0.1$, $w_0 = 1000$, and $w_1 = 18{,}000$. Also,

$$f_0(x) = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2}(x-50)^2\right]$$

and

$$f_1(x) = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2}(x-52)^2\right].$$

By the results of this section, it should be decided that the process is out of control if

$$\frac{f_1(x)}{f_0(x)} > \frac{\xi_0 w_0}{\xi_1 w_1} = \frac{1}{2}.$$

This inequality can be reduced to the inequality $2x - 102 > -\log\ 2$ or, equivalently, $x > 50.653$.

4. In this exercise, $\xi_0 = 1/4$, $\xi_1 = 3/4$, and $w_0 = w_1 = 1$. Let $x_1, \ldots, x_n$ denote the observed values in the sample, and let $y = \sum_{i=1}^{n} x_i$. Then

$$f_0(\boldsymbol{X}) = (0.3)^y (0.7)^{n-y}$$

and

$$f_1(\boldsymbol{X}) = (0.4)^y (0.6)^{n-y}.$$

By the results of this section, H_0 should be rejected if

$$\frac{f_1(\boldsymbol{X})}{f_0(\boldsymbol{X})} > \frac{\xi_0 w_0}{\xi_1 w_1} = \frac{1}{3}.$$

But

$$\frac{f_1(\boldsymbol{X})}{f_0(\boldsymbol{X})} = \left(\frac{4}{3}\cdot\frac{7}{6}\right)^y \left(\frac{6}{7}\right)^n > \frac{1}{3}$$

if and only if

$$y\ \log\frac{14}{9} + n\ \log\frac{6}{7} > \log\frac{1}{3}$$

or, equivalently, if and only if

$$\bar{x}_n = \frac{y}{n} > \frac{\log\dfrac{7}{6} + \dfrac{1}{n}\log\dfrac{1}{3}}{\log\dfrac{14}{9}}.$$

8.9 Solutions to Exercises

2. When $\mu = 0$, $100\overline{X}_n$ has a standard normal distribution. Therefore, $\Pr(100\,|\overline{X}_n| > 1.96\,|\,\mu = 0) = 0.05$. It follows that $c = 1.96/100 = 0.0196$.

   (a) When $\mu = 0.01$, the random variable $Z = 100(\overline{X}_n - 0.01)$ has a standard normal distribution. Therefore,

   $$\begin{aligned}\Pr(|\overline{X}_n| < c\,|\,\mu = 0.01) &= \Pr(-1.96 < 100\overline{X}_n < 1.96\,|\,\mu = 0.01)\\ &= \Pr(-2.96 < z < 0.96)\\ &= 0.8315 - 0.0015 = 0.8300.\end{aligned}$$

   It follows that $\Pr(|\overline{X}_n| \geq c\,|\,\mu = 0.01) = 1 - 0.8300 = 0.1700$.

   (b) When $\mu = 0.02$, the random variable $Z = 100(\overline{X}_n - 0.02)$ has a standard normal distribution. Therefore,

   $$\begin{aligned}\Pr(|\overline{X}_n| < c\,|\,\mu = 0.02) &= \Pr(-3.96 < Z < -0.04)\\ &= \Pr(0.04 < Z < 3.96) = 1 - 0.5160.\end{aligned}$$

   It follows that $\Pr(|\overline{X}_n| < c\,|\,\mu = 0.02) = 0.5160$.

18. If U is defined as in Eq. (7.6.9), then the prior distribution of U is a t distribution with $2\alpha_0 = 2$ degrees of freedom. Since the t distribution is symmetric with respect to the origin, it follows that under the prior distribution, $\Pr(H_0) = \Pr(\mu \leq 3) = \Pr(U \leq 0) = 1/2$. It follows from (7.6.1) and (7.6.2) that under the posterior distribution,

$$\begin{aligned}\mu_1 &= \frac{3 + (17)(3.2)}{1 + 17} = 3.189\,, \quad \lambda_1 = 18\,,\\ \alpha_1 &= 1 + \frac{17}{2} = 9.5\,,\\ \beta_1 &= 1 + \frac{1}{2}(17) + \frac{(17)(.04)}{2(18)} = 9.519.\end{aligned}$$

If we now define Y to be the random variable in Eq. (7.6.12) then $Y = (4.24)(\mu - 3.19)$ and Y has a t distribution with $2\alpha_1 = 19$ degrees of freedom. Thus, under the posterior distribution,

$$\Pr(H_0) = \Pr(\mu \leq 3) = \Pr[Y \leq (4.24)(3 - 3.19)] = \Pr(Y \leq -.81) = \Pr(Y \geq .81).$$

It is found from the table of the t distribution with 19 degrees of freedom that this probability is approximately 0.21.