Chapter 1

Prerequisites

The authors would like nothing more than to dive right into the sheer excitement of Precalculus. However, experience - our own as well as that of our colleagues - has taught us that is it beneficial, if not completely necessary, to review what students should know before embarking on a Precalculus adventure. The goal of Chapter **??** is exactly that: to review the concepts, skills and vocabulary we believe are prerequisite to a rigorous, college-level Precalculus course. This review is not designed to teach the material to students who have never seen it before thus the presentation is more succinct and the exercise sets are shorter than those usually found in an Intermediate Algebra text. An outline of the chapter is given below.

Section 1.1 (Basic Set Theory and Interval Notation) contains a brief summary of the set theory terminology used throughout the text including sets of real numbers and interval notation.

Section 1.2 (Real Number Arithmetic) lists the properties of real number arithmetic.[1]

Section 1.3 (Linear Equations and Inequalities) focuses on solving linear equations and linear inequalities from a strictly algebraic perspective. The geometry of graphing lines in the plane is deferred until Section **??** (Linear Functions).

Section 1.4 (Absolute Value Equations and Inequalities) begins with a definition of absolute value as a distance. Fundamental properties of absolute value are listed and then basic equations and inequalities involving absolute value are solved using the 'distance definition' and those properties. Absolute value is revisited in much greater depth in Section **??** (Absolute Value Functions).

Section 1.5 (Polynomial Arithmetic) covers the addition, subtraction, multiplication and division of polynomials as well as the vocabulary which is used extensively when the graphs of polynomials are studied in Chapter **??** (Polynomials).

Section 1.6 (Factoring) covers basic factoring techniques and how to solve equations using those techniques along with the Zero Product Property of Real Numbers.

Section 1.7 (Quadratic Equations) discusses solving quadratic equations using the technique of 'completing the square' and by using the Quadratic Formula. Equations which are 'quadratic in form' are also discussed.

Section 1.8 (Rational Expressions and Equations) starts with the basic arithmetic of rational expressions and the simplifying of compound fractions. Solving equations by clearing denominators and the handling negative integer exponents are presented but the graphing of rational functions is deferred until Chapter **??** (Rational Functions).

Section 1.9 (Radicals and Equations) covers simplifying radicals as well as the solving of basic equations involving radicals.

Section 1.10 (Complex Numbers) covers the basic arithmetic of complex numbers and the solving of quadratic equations with complex solutions.

[1] You know, the stuff students mess up all of the time like fractions and negative signs. The collection is close to exhaustive and definitely exhausting!

1.1 Basic Set Theory and Interval Notation

1.1.1 Some Basic Set Theory Notions

Like all good Math books, we begin with a definition.

Definition 1.1. A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

The collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1] In general, there are three ways to describe sets and those methods are listed below.

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define the set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "smolko". A **roster** description of S is $\{s, m, o, l, k\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the *order* of the elements doesn't matter, so $\{k, l, m, o, s\}$ is also a roster description of S. Moving right along, a **set-builder** description of S is: $\{x \mid x \text{ is a letter in the word "smolko"}\}$. The way to read this is 'The set of elements x such that x is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word "smolko"}\}$.

Notice that m is in S but many other letters, such as q, are not in S. We express these ideas of set inclusion and exclusion mathematically using the symbols $m \in S$ (read 'm is in S') and $q \notin S$ (read 'q is not in S'). More precisely, we have the following.

Definition 1.2. Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "smolko"}\}$. A roster description of C is $C = \{s, m, l, k\}$. Note that by construction, every element of C is also in S. We express

[1] For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous Russell's Paradox.

this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 1.3. Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{s, m, o, l, k, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 1.4. Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 1.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

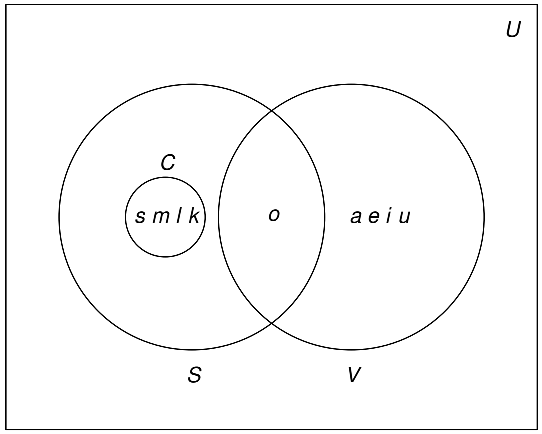
Returning to the sets C and V above, $C \cup V = \{s, m, l, k, a, e, i, o, u\}$.[2] When it comes to their intersection, however, we run into a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 1.5. The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance.[4] Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn at the top of the next page.

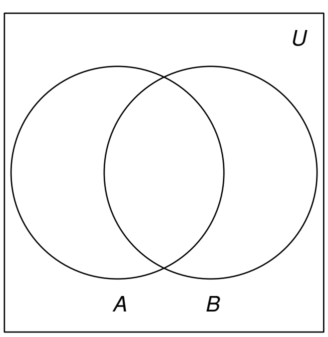
[2] Which just so happens to be the same set as $S \cup V$.
[3] Sadly, the full extent of the empty set's role will not be explored in this text.
[4] Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!



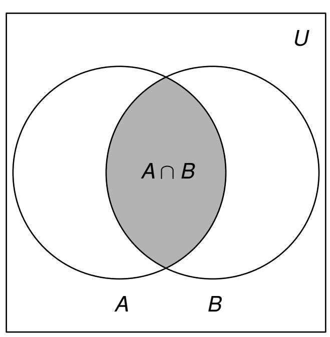
A Venn Diagram for C, S and V.

In the Venn Diagram above we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

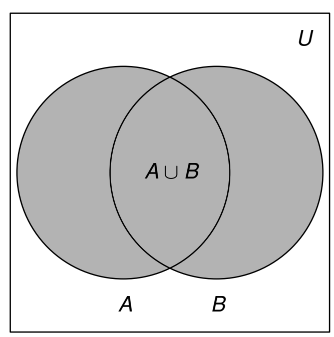
All of these circles lie in a rectangle labeled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

1.1.2 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[5] definition of a real number is given below.

Definition 1.6. A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[6] the real numbers are *constructed* from some of these subsets.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis '...' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.
2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]
4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers[b] is:

$$\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$$

5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$.[c] That is, an irrational number is a real number which isn't rational. Said differently,

$$\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$$

[a] The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.
[b] See Section **??**.
[c] Examples here include number π (See Section **??**), $\sqrt{2}$ and $0.101001000100001\ldots$.

Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and since every rational number is a real number[7] the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above. It is time for an example.

[5] Math pun intended!
[6] See, for instance, Landau's Foundations of Analysis.
[7] Thanks to long division!

Example 1.1.1.

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.
3. Let $A = \{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\}$.
   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?
   (b) Find $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.
4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

Solution.

1. To find a roster description for these sets, we need to list their elements. Starting with $P = \{2^n \mid n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '...' indicates the that pattern continues.[8]

   Proceeding in the same way, we generate elements in $E = \{2n \mid n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers.[9] A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper.[10] What makes the set $S = \{x^2 \mid x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[11] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything

[8] This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[9] This shouldn't be too surprising, since an even integer is *defined* to be an integer multiple of 2.

[10] Think of this as an opportunity to stop and smell the mathematical roses.

[11] This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

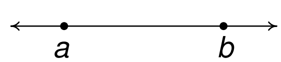
in S is a non-negative real number.[12] This begs the question: are all of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number.[13] Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. (a) The set A contains no natural numbers.[14] Clearly, $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[15] As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

   (b) The set $A \cap \mathbb{W} = \{x \,|\, x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in A which are whole numbers. Since A contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in A which are integers. Since -117 is the only integer in A, $A \cap \mathbb{Z} = \{-117\}$. As for the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, so $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[16] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. □

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown below.



The real number line with two numbers a and b where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there

[12] This means S is a subset of the non-negative real numbers.
[13] This is called the 'square root closed' property of the non-negative real numbers.
[14] Carl was tempted to include $0.\overline{9}$ in the set A, but thought better of it. See Section **??** for details.
[15] So $0.20\overline{2002} = 0.20200220022002\ldots$.
[16] In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

are no 'holes' or 'gaps' in the real number line.[17] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

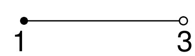
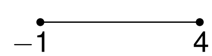
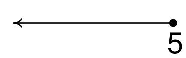
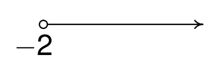
> **Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$**
> Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. ('Try' is the operative word, here.[18])

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the 'Law of Trichotomy'.

> **Law of Trichotomy**
> If a and b are real numbers then exactly one of the following statements is true:
> $a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This

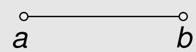
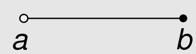
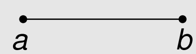
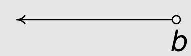
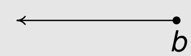
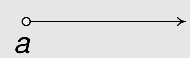
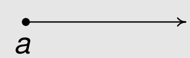
[17]Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.

[18]Again, see Section **??** for details.

corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[19]

Interval Notation

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

We close this section with an example that ties together several concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of 'union' and 'intersection' to describe a variety of sets on the real number line.

Example 1.1.2.

[19] The importance of understanding interval notation in Calculus cannot be overstated so please do yourself a favor and memorize this chart.

1. Express the following sets of numbers using interval notation.

   (a) $\{x \mid x \leq -2 \text{ or } x \geq 2\}$

   (b) $\{x \mid x \neq 3\}$

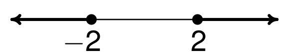
   (c) $\{x \mid x \neq \pm 3\}$

   (d) $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

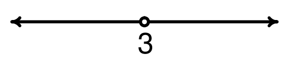
Solution.

1. (a) The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.



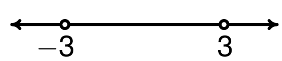
$(-\infty, -2] \cup [2, \infty)$

   (b) For the set $\{x \mid x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in one of these intervals *or* the other, we once again use the union symbol to get $\{x \mid x \neq 3\} = (-\infty, 3) \cup (3, \infty)$.



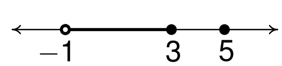
$(-\infty, 3) \cup (3, \infty)$

   (c) For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Do you remember what we said back on 6 about $x = \pm 3$?) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.



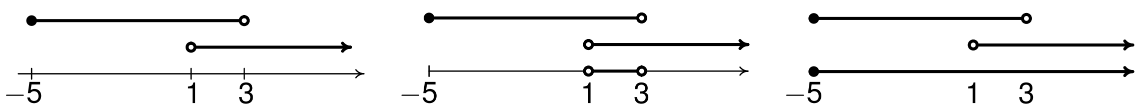
$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

   (d) Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1, 3]$ along with the single number 5. While we *could* express this single point as $[5, 5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. Thus our final answer is $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.



$(-1, 3] \cup \{5\}$

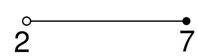
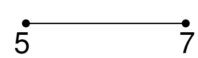
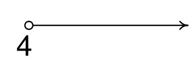
2. We start by graphing $A = [-5, 3)$ and $B = (1, \infty)$ on the number line. To find $A \cap B$, we need to find the numbers in common to both A and B, in other words, the overlap of the two intervals. Clearly, everything between 1 and 3 is in both A and B. However, since 1 is in A but not in B, 1 is not in the intersection. Similarly, since 3 is in B but not in A, it isn't in the intersection either. Hence, $A \cap B = (1, 3)$. To find $A \cup B$, we need to find the numbers in at least one of A or B. Graphically, we shade A and B along with it. Notice here that even though 1 isn't in B, it is in A, so it's the union along with all the other elements of A between -5 and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both A and B together gives us $A \cup B = [-5, \infty)$.



$A = [-5, 3)$, $B = (1, \infty)$ $A \cap B = (1, 3)$ $A \cup B = [-5, \infty)$ □

1.1.3 Exercises

1. Find a verbal description for $O = \{2n - 1 \mid n \in \mathbb{N}\}$
2. Find a roster description for $X = \{z^2 \mid z \in \mathbb{Z}\}$
3. Let $A = \left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \frac{20}{10}, 117\right\}$
   (a) List the elements of A which are natural numbers.
   (b) List the elements of A which are irrational numbers.
   (c) Find $A \cap \mathbb{Z}$
   (d) Find $A \cap \mathbb{Q}$
4. Fill in the chart below.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 5 - 10, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1,5] \cap [0,8)$

6. $(-1,1) \cup [0,6]$

7. $(-\infty,4] \cap (0,\infty)$

8. $(-\infty,0) \cap [1,5]$

9. $(-\infty,0) \cup [1,5]$

10. $(-\infty,5] \cap [5,8)$

In Exercises 11 - 22, write the set using interval notation.

11. $\{x \mid x \neq 5\}$

12. $\{x \mid x \neq -1\}$

13. $\{x \mid x \neq -3,\ 4\}$

14. $\{x \mid x \neq 0,\ 2\}$

15. $\{x \mid x \neq 2,\ -2\}$

16. $\{x \mid x \neq 0,\ \pm 4\}$

17. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

18. $\{x \mid x < 3 \text{ or } x \geq 2\}$

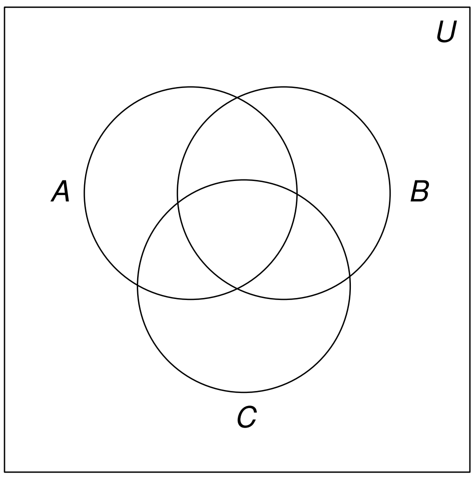
19. $\{x \mid x \leq -3 \text{ or } x > 0\}$

20. $\{x \mid x \leq 5 \text{ or } x = 6\}$

21. $\{x \mid x > 2 \text{ or } x = \pm 1\}$

22. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

For Exercises 23 - 28, use the blank Venn Diagram below A, B, and C as a guide for you to shade the following sets.



23. $A \cup C$

24. $B \cap C$

25. $(A \cup B) \cup C$

26. $(A \cap B) \cap C$

27. $A \cap (B \cup C)$

28. $(A \cap B) \cup (A \cap C)$

29. Explain how your answers to problems 27 and 28 show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.

1.1.4 Answers

1. O is the odd natural numbers.

2. $X = \{0, 1, 4, 9, 16, ...\}$

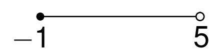
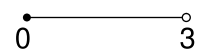
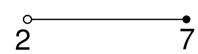
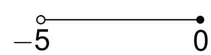
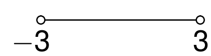
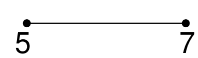
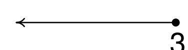
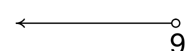
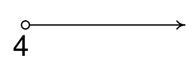
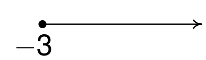
3. (a) $\frac{20}{10} = 2$ and 117

   (b) $\sqrt{3}$ and 5.2020020002

   (c) $\left\{-3, \frac{20}{10}, 117\right\}$

   (d) $\left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \frac{20}{10}, 117\right\}$

4.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

5. $(-1, 5] \cap [0, 8) = [0, 5]$

6. $(-1, 1) \cup [0, 6] = (-1, 6]$

7. $(-\infty, 4] \cap (0, \infty) = (0, 4]$

8. $(-\infty, 0) \cap [1, 5] = \emptyset$

9. $(-\infty, 0) \cup [1, 5] = (-\infty, 0) \cup [1, 5]$

10. $(-\infty, 5] \cap [5, 8) = \{5\}$

11. $(-\infty, 5) \cup (5, \infty)$

12. $(-\infty, -1) \cup (-1, \infty)$

13. $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

14. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

15. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

16. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$

17. $(-\infty, -1] \cup [1, \infty)$

18. $(-\infty, \infty)$

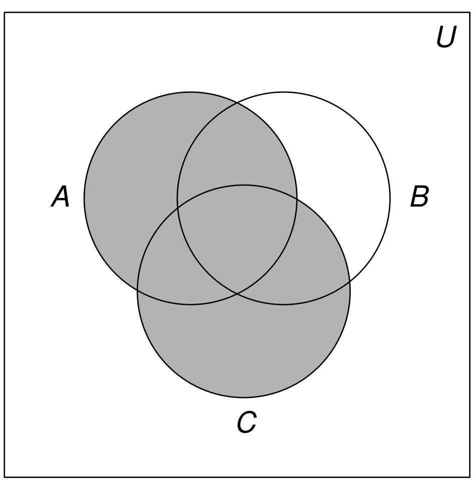
19. $(-\infty, -3] \cup (0, \infty)$

20. $(-\infty, 5] \cup \{6\}$

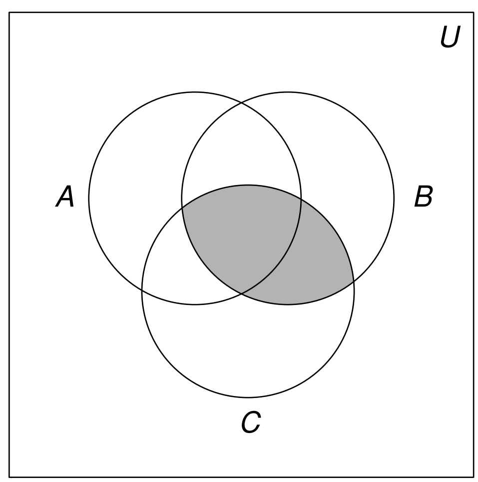
21. $\{-1\} \cup \{1\} \cup (2, \infty)$

22. $(-3, 3) \cup \{4\}$

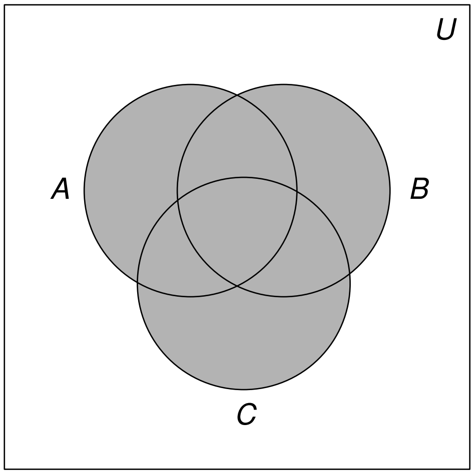
23. $A \cup C$



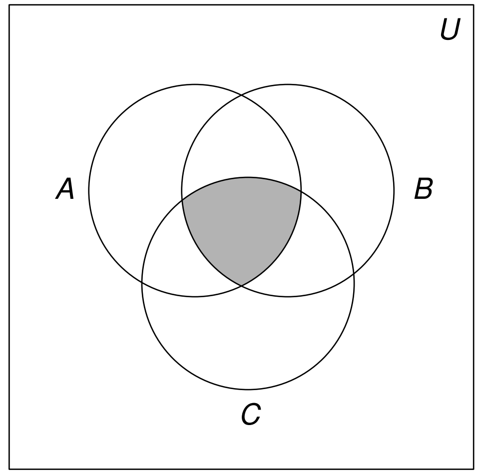
24. $B \cap C$



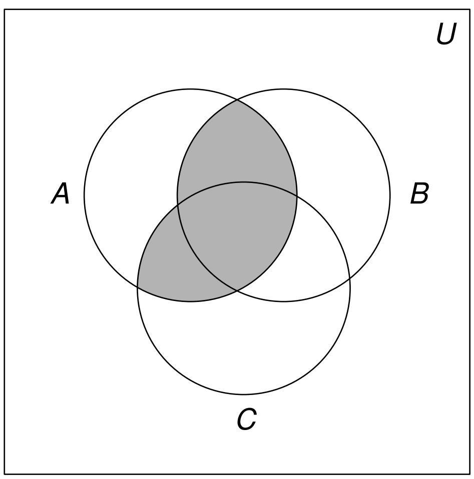
25. $(A \cup B) \cup C$



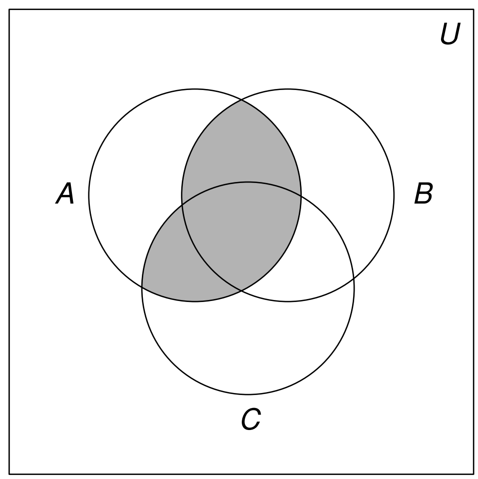
26. $(A \cap B) \cap C$



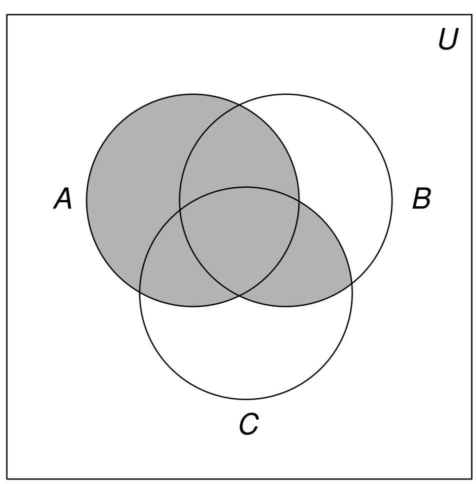
27. $A \cap (B \cup C)$



28. $(A \cap B) \cup (A \cap C)$



29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception in Definition 1.7.

Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right)$.

While most students and some faculty tend to skip over these properties or give them a cursory glance at best,[1] it is important to realize that the properties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers, $1 + 2 + 3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1 + 2) + 3$ or $1 + (2 + 3)$. This brings up a

[1] Not unlike how Carl approached all the Elven poetry in The Lord of the Rings.

note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
- **Factoring:**[a] $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

[a] Or, as Carl calls it, 'reading the Distributive Property from right to left.'

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b + c)$ (distributing from the left) and $(a + b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both)

  Note: This not only says that $0 \cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0 \cdot \left(\dfrac{1}{a}\right) = 0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.[a]

[a] The expression $\frac{0}{0}$ is technically an 'indeterminant form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter **??**.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2 + x - 6 = 0$ is by factoring[2] the left hand side of this equation to get $(x - 2)(x + 3) = 0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x - 2 = 0$ or $x + 3 = 0$ so $x = 2$ or $x = -3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter **??**.

Next up is a review of the arithmetic of 'negatives'. On page 18 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5 - 3 = 5 + (-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5-3$' and '-3' even though they are mathematically quite different.[3] In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key.

Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:**[a] $-a - b = -(a + b)$ and $b - a = -(a - b)$.

[a]Or, as Carl calls it, reading 'Distributing' Negatives from right to left.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a + b) = (-1)(a + b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$. Negatives do not 'distribute' across multiplication: $-(2 \cdot 3) \neq (-2) \cdot (-3)$. Instead, $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$. The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$. Speaking of fractions, we now review their arithmetic.

[2]Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[3]We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \frac{a}{1}$ and $\frac{a}{a} = 1$.
- **Fraction Equality:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In particular: $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$
  Note: A common denominator is **not** required to **multiply** fractions!
- **Division[a] of Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.
  In particular: $1 \div \frac{a}{b} = \frac{b}{a}$ and $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$
  Note: A common denominator is **not** required to **divide** fractions!
- **Addition and Subtraction of Fractions:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$.
  Note: A common denominator **is** required to **add or subtract** fractions!
- **Equivalent Fractions:** $\frac{a}{b} = \frac{ad}{bd}$, since $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$
  Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.
- **'Reducing'[b] Fractions:** $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$, since $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$.
  In particular, $\frac{ab}{b} = a$ since $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$ and $\frac{b-a}{a-b} = \frac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.
  Note: We may only cancel common **factors** from both numerator and denominator.

[a]The old 'invert and multiply' or 'fraction gymnastics' play.
[b]Or 'Canceling' Common Factors - this is really just reading the previous property 'from right to left'.

Students make so many mistakes with fractions that we feel it is necessary to pause a moment in the narrative and offer you the following example.

Example 1.2.1. Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4} + \dfrac{6}{7}$
2. $\dfrac{5}{12} - \left(\dfrac{47}{30} - \dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{7}{3-5} - \dfrac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\dfrac{12}{5} - \dfrac{7}{24}}{1 + \left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3)) - 5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right) - \left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution.

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned} \frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{Equivalent Fractions} \\ &= \frac{7}{28} + \frac{24}{28} && \text{Multiplication of Fractions} \\ &= \frac{31}{28} && \text{Addition of Fractions} \end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30} - \frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[4] The lowest common denominator[5] for all three fractions is 60.

$$\begin{aligned} \frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{Equivalent Fractions} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\ &= \frac{71}{60} && \text{Addition and Subtraction of Fractions} \end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

[4] See the remark on page 18 about how we add $1 + 2 + 3$.

[5] We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction.[6] The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\frac{7}{3-5} - \frac{7}{3-5.21}}{5-5.21} &= \frac{\frac{7}{-2} - \frac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\frac{7}{2} + \frac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\frac{7}{2} - \frac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (since $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\frac{7}{2} - \frac{7}{2.21}}{0.21} = \frac{\frac{7}{2} - \frac{7 \cdot 100}{2.21 \cdot 100}}{0.21} = \frac{\frac{7}{2} - \frac{700}{221}}{0.21}$$

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' trick and then cancel any factors that we can.

$$\begin{aligned}
\frac{\frac{7}{2} - \frac{700}{221}}{0.21} &= \frac{\frac{7}{2} \cdot \frac{221}{221} - \frac{700}{221} \cdot \frac{2}{2}}{\frac{21}{100}} = \frac{\frac{1547}{442} - \frac{1400}{442}}{\frac{21}{100}} \\
&= \frac{\frac{147}{442}}{\frac{21}{100}} = \frac{147}{442} \cdot \frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42 \cdot 350$ and $9282 = 42 \cdot 221$.

[6] Fractionception, perhaps?

[7] Also called a 'vinculum'.

[8] Try it if you don't believe us.

4. We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the numerator from the denominator.

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' trick. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} - \frac{7}{24}\right) \cdot 120}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right) \cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120) - \left(\frac{7}{24}\right)(120)}{(1)(120) + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
&= \frac{\frac{12 \cdot 120}{5} - \frac{7 \cdot 120}{24}}{120 + \frac{12 \cdot 7 \cdot 120}{5 \cdot 24}} && \text{Multiply fractions} \\
&= \frac{\frac{12 \cdot 24 \cdot \cancel{5}}{\cancel{5}} - \frac{7 \cdot 5 \cdot \cancel{24}}{\cancel{24}}}{120 + \frac{12 \cdot 7 \cdot \cancel{5} \cdot \cancel{24}}{\cancel{5} \cdot \cancel{24}}} && \text{Factor and cancel} \\
&= \frac{(12 \cdot 24) - (7 \cdot 5)}{120 + (12 \cdot 7)} \\
&= \frac{288 - 35}{120 + 84} \\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

5. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned} \frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\ &= \frac{(5)(0)+15}{-2} \\ &= \frac{15}{-2} \\ &= -\frac{15}{2} && \text{Properties of Negatives} \end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

6. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned} \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\ &= \frac{15}{65}-\frac{-48}{65} \\ &= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\ &= \frac{15+48}{65} && \text{Add numerators} \\ &= \frac{63}{65} \end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a]See the comment regarding '$\frac{0}{0}$' on page 19.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2 b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a+b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols.)
2. Evaluate **e**xponents.
3. Evaluate **m**ultiplication and **d**ivision as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

We note that there are many useful mnemonic device for remembering the order of operations.[a]

[a] Our favorite is '**P**lease **e**ntertain **m**y **d**ear **a**uld **S**asquatch.'

For example, $2 + 3 \cdot 4^2 = 2 + 3 \cdot 16 = 2 + 48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0 - 3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0 - 3^2 = 0 - 9 = -9$. In general, we interpret $-a^n = -(a^n)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2 + x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172} \cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

Example 1.2.2. Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution.

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 1.2.1, the division here acts as a grouping symbol, so we save the division to the end.

$$\begin{aligned}\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4} \\ &= \frac{16-16}{4} = \frac{0}{4} = 0\end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5} &= 12(-5)(-2)^{-4}+6(-5)^2(-4)(-2)^{-5} \\ &= 12(-5)\left(\frac{1}{(-2)^4}\right)+6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\ &= 12(-5)\left(\frac{1}{16}\right)+6(25)(-4)\left(\frac{1}{-32}\right) \\ &= (-60)\left(\frac{1}{16}\right)+(-600)\left(\frac{1}{-32}\right) \\ &= \frac{-60}{16}+\left(\frac{-600}{-32}\right) \\ &= \frac{-15\cdot \cancel{4}}{4\cdot \cancel{4}}+\frac{-75\cdot \cancel{8}}{-4\cdot \cancel{8}} \\ &= \frac{-15}{4}+\frac{-75}{-4} \\ &= \frac{-15}{4}+\frac{75}{4} \\ &= \frac{-15+75}{4} \\ &= \frac{60}{4} \\ &= 15\end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular)

to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\frac{5 \cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5 \cdot 3^{49}}{4^{34}}\right)} &= \frac{5 \cdot 3^{51}}{4^{36}} \cdot \frac{4^{34}}{5 \cdot 3^{49}} &= \frac{\cancel{5} \cdot 3^{51} \cdot 4^{34}}{\cancel{5} \cdot 3^{49} \cdot 4^{36}} &= \frac{3^{51}}{3^{49}} \cdot \frac{4^{34}}{4^{36}} \\
&= 3^{51-49} \cdot 4^{34-36} &= 3^2 \cdot 4^{-2} &= 3^2 \cdot \left(\frac{1}{4^2}\right) \\
&= 9 \cdot \left(\frac{1}{16}\right) &= \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 1.2.1. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} &= \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^{2}} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} \\
&= \frac{\left(\frac{24}{5}\right)\cdot 25}{\left(1-\frac{144}{25}\right)\cdot 25} = \frac{\left(\frac{24 \cdot 5 \cdot \cancel{5}}{\cancel{5}}\right)}{\left(1 \cdot 25 - \frac{144 \cdot \cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Since 120 and 119 have no common factors, we are done. □

One of the places where the properties of exponents play an important role is in the use of **Scientific Notation**. The basis for scientific notation is that since we use decimals (base ten numerals) to represent real numbers, we can adjust where the decimal point lies by multiplying by an appropriate power of 10. This allows scientists and engineers to focus in on the 'significant' digits[9] of a number - the nonzero values - and adjust for the decimal places later. For instance, $-621 = -6.21 \times 10^2$ and $0.023 = 2.3 \times 10^{-2}$. Notice here that we revert to using the familiar '$\times$' to indicate multiplication.[10] In general, we arrange the real number so exactly one non-zero digit appears to the left of the decimal point. We make this idea precise in the following:

[9] Awesome pun!

[10] This is the 'notable exception' we alluded to earlier.

Definition 1.7. A real number is written in **Scientific Notation** if it has the form $\pm n.d_1 d_2 \ldots \times 10^k$ where n is a natural number, d_1, d_2, etc., are whole numbers, and k is an integer.

On calculators, scientific notation may appear using an 'E' or 'EE' as opposed to the $\times$ symbol. For instance, while we will write 6.02×10^{23} in the text, the calculator may display 6.02 E 23 or 6.02 EE 23.

Example 1.2.3. Perform the indicated operations and simplify. Write your final answer in scientific notation, rounded to two decimal places.

1. $\dfrac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}}$
2. $\left(2.13 \times 10^{53}\right)^{100}$

Solution.

1. As mentioned earlier, the point of scientific notation is to separate out the 'significant' parts of a calculation and deal with the powers of 10 later. In that spirit, we separate out the powers of 10 in both the numerator and the denominator and proceed as follows

$$\begin{aligned}
\frac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}} &= \frac{(6.626)(3.14)}{1.78} \cdot \frac{10^{-34} \cdot 10^{9}}{10^{23}} \\
&= \frac{20.80564}{1.78} \cdot \frac{10^{-34+9}}{10^{23}} \\
&= 11.685\ldots \cdot \frac{10^{-25}}{10^{23}} \\
&= 11.685\ldots \times 10^{-25-23} \\
&= 11.685\ldots \times 10^{-48}
\end{aligned}$$

   We are asked to write our final answer in scientific notation, rounded to two decimal places. To do this, we note that $11.685\ldots = 1.1685\ldots \times 10^{1}$, so

$$11.685\ldots \times 10^{-48} = 1.1685\ldots \times 10^{1} \times 10^{-48} = 1.1685\ldots \times 10^{1-48} = 1.1685\ldots \times 10^{-47}$$

   Our final answer, rounded to two decimal places, is 1.17×10^{-47}.

   We could have done that whole computation on a calculator so why did we bother doing any of this by hand in the first place? The answer lies in the next example.

2. If you try to compute $\left(2.13 \times 10^{53}\right)^{100}$ using most hand-held calculators, you'll most likely get an 'overflow' error. It is possible, however, to use the calculator in combination with the properties of exponents to compute this number. Using properties of exponents, we get:

$$\begin{aligned}
\left(2.13 \times 10^{53}\right)^{100} &= (2.13)^{100}\left(10^{53}\right)^{100} \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{53\times 100}\right) \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{5300}\right) \\
&= 6.885\ldots \times 10^{32} \cdot 10^{5300} \\
&= 6.885\ldots \times 10^{5332}
\end{aligned}$$

To two decimal places our answer is 6.88×10^{5332}. □

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 1.8. Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 1.8 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[11] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

Theorem 1.1. Properties of Radicals: Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 1.1 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is

[11] See Chapter **??**.

proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with.[12] We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 1.9. Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number.[a]
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

[a] If n is even we need $a \geq 0$.

It would make life really nice if the rational exponents defined in Definition 1.9 had all of the same properties that integer exponents have as listed on page 26 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 1.9, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.[13]

Example 1.2.4. Perform the indicated operations and simplify.

[12] Otherwise we'd run into an interesting paradox. See Section 1.10.

[13] Much to Jeff's chagrin. He's fairly traditional and therefore doesn't care much for radicals.

1. $\dfrac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1-\left(\dfrac{\sqrt{3}}{3}\right)^2}$

3. $(\sqrt[3]{-2}-\sqrt[3]{-54})^2$

4. $2\left(\dfrac{9}{4}-3\right)^{1/3}+2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4}-3\right)^{-2/3}$

Solution.

1. We begin in the numerator and note that the radical here acts a grouping symbol,[14] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} &= \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)} \\
&= \frac{-(-4)-\sqrt{16+24}}{2(2)} \\
&= \frac{-(-4)-\sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)} \\
&= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}
\end{aligned}$$

Since the numerator and denominator have no more common factors,[15] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denomi-

[14] The line extending horizontally from the square root symbol '$\sqrt{\ }$' is, you guessed it, another vinculum.

[15] Do you see why we aren't 'canceling' the remaining 2's?

nator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot \cancel{3}}{3\cdot \cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[16] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, -1 *is* a perfect cube, and we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2} + 3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

[16] Of an integer, that is!

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

□

1.2.1 Exercises

In Exercises 1 - 33, perform the indicated operations and simplify.

1. $5-2+3$
2. $5-(2+3)$
3. $\frac{2}{3}-\frac{4}{7}$
4. $\frac{3}{8}+\frac{5}{12}$
5. $\frac{5-3}{-2-4}$
6. $\frac{2(-3)}{3-(-3)}$
7. $\frac{2(3)-(4-1)}{2^2+1}$
8. $\frac{4-5.8}{2-2.1}$
9. $\frac{1-2(-3)}{5(-3)+7}$
10. $\frac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\frac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\frac{(-2)^2-(-2)-6}{(-2)^2-4}$
13. $\frac{3-\frac{4}{9}}{-2-(-3)}$
14. $\frac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\frac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\frac{2}{3}\right)^{-5}$
18. $3^{-1}-4^{-2}$
19. $\frac{1+2^{-3}}{3-4^{-1}}$
20. $\frac{3\cdot 5^{100}}{12\cdot 5^{98}}$
21. $\sqrt{3^2+4^2}$
22. $\sqrt{12}-\sqrt{75}$
23. $(-8)^{2/3}-9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3-4)^2+(5-2)^2}$
26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
28. $\frac{-12+\sqrt{18}}{21}$
29. $\frac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
30. $\frac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
31. $2(-5)(-5+1)^{-1}+(-5)^2(-1)(-5+1)^{-2}$
32. $3\sqrt{2(4)+1}+3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1-(-7)}+(-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$
34. With the help of your calculator, find $(3.14\times 10^{87})^{117}$. Write your final answer, using scientific notation, rounded to two decimal places. (See Example 1.2.3.)

1.2.2 Answers

1. 6
2. 0
3. $\frac{2}{21}$
4. $\frac{19}{24}$
5. $-\frac{1}{3}$
6. -1
7. $\frac{3}{5}$
8. 18
9. $-\frac{7}{8}$
10. Undefined.
11. 0
12. Undefined.
13. $\frac{23}{9}$
14. $-\frac{4}{99}$
15. $-\frac{24}{7}$
16. 0
17. $\frac{243}{32}$
18. $\frac{13}{48}$
19. $\frac{9}{22}$
20. $\frac{25}{4}$
21. 5
22. $-3\sqrt{3}$
23. $\frac{107}{27}$
24. $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$
25. $\sqrt{10}$
26. $\frac{\sqrt{61}}{2}$
27. $\sqrt{7}$
28. $\frac{-4+\sqrt{2}}{7}$
29. -1
30. $2+\sqrt{5}$
31. $\frac{15}{16}$
32. 13
33. $-\frac{385}{12}$
34. 1.38×10^{10237}

1.3 Linear Equations and Inequalities

In the introduction to this chapter we said that we were going to review "the concepts, skills and vocabulary we believe are prerequisite to a rigorous, college-level Precalculus course." So far, we've presented a lot of vocabulary and concepts but we haven't done much to refresh the skills needed to survive in the Precalculus wilderness. Thus over the course of the next few sections we will focus our review on the Algebra skills needed to solve basic equations and inequalities. In general, equations and inequalities fall into one of three categories: conditional, identity or contradiction, depending on the nature of their solutions. A **conditional** equation or inequality is true for only *certain* real numbers. For example, $2x + 1 = 7$ is true precisely when $x = 3$, and $w - 3 \leq 4$ is true precisely when $w \leq 7$. An **identity** is an equation or inequality that is true for *all* real numbers. For example, $2x - 3 = 1 + x - 4 + x$ or $2t \leq 2t + 3$. A **contradiction** is an equation or inequality that is *never* true. Examples here include $3x - 4 = 3x + 7$ and $a - 1 > a + 3$.

As you may recall, solving an equation or inequality means finding all of the values of the variable, if any exist, which make the given equation or inequality true. This often requires us to manipulate the given equation or inequality from its given form to an easier form. For example, if we're asked to solve $3 - 2(x - 3) = 7x + 3(x + 1)$, we get $x = \frac{1}{2}$, but not without a fair amount of algebraic manipulation. In order to obtain the correct answer(s), however, we need to make sure that whatever maneuvers we apply are reversible in order to guarantee that we maintain a chain of **equivalent** equations or inequalities. Two equations or inequalities are called **equivalent** if they have the same solutions. We list these 'legal moves' below.

Procedures which Generate Equivalent Equations

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **nonzero** real number.[a]

Procedures which Generate Equivalent Inequalities

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **positive** real number.[b]

[a]Multiplying both sides of an equation by 0 collapses the equation to $0 = 0$, which doesn't do anybody any good.

[b]Remember that if you multiply both sides of an inequality by a negative real number, the inequality sign is reversed: $3 \leq 4$, but $(-2)(3) \geq (-2)(4)$.

1.3.1 Linear Equations

The first type of equations we need to review are **linear** equations as defined below.

Definition 1.10. An equation is said to be **linear** in a variable X if it can be written in the form $AX = B$ where A and B are expressions which do not involve X and $A \neq 0$.

One key point about Definition 1.10 is that the exponent on the unknown 'X' in the equation is 1, that is $X = X^1$. Our main strategy for solving linear equations is summarized below.

Strategy for Solving Linear Equations

In order to solve an equation which is linear in a given variable, say X:

1. Isolate all of the terms containing X on one side of the equation, putting all of the terms not containing X on the other side of the equation.
2. Factor out the X and divide both sides of the equation by its coefficient.

We illustrate this process with a collection of examples below.

Example 1.3.1. Solve the following equations for the indicated variable. Check your answer.

1. Solve for x: $3x - 6 = 7x + 4$
2. Solve for t: $3 - 1.7t = \frac{t}{4}$
3. Solve for a: $\frac{1}{18}(7 - 4a) + 2 = \frac{a}{3} - \frac{4-a}{12}$
4. Solve for y: $8y\sqrt{3} + 1 = 7 - \sqrt{12}(5 - y)$
5. Solve for x: $\frac{3x-1}{2} = x\sqrt{50} + 4$
6. Solve for y: $x(4 - y) = 8y$

Solution.

1. The variable we are asked to solve for is x so our first move is to gather all of the terms involving x on one side and put the remaining terms on the other.[1]

$$\begin{array}{rcll} 3x - 6 & = & 7x + 4 & \\ (3x - 6) - 7x + 6 & = & (7x + 4) - 7x + 6 & \text{Subtract } 7x\text{, add } 6 \\ 3x - 7x - 6 + 6 & = & 7x - 7x + 4 + 6 & \text{Rearrange terms} \\ -4x & = & 10 & 3x - 7x = (3 - 7)x = -4x \\ \frac{-4x}{-4} & = & \frac{10}{-4} & \text{Divide by the coefficient of } x \\ x & = & -\frac{5}{2} & \text{Reduce to lowest terms} \end{array}$$

To check our answer, we substitute $x = -\frac{5}{2}$ into each side of the **orginial** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right) - 6 = -\frac{27}{2}$ and $7\left(-\frac{5}{2}\right) + 4 = -\frac{27}{2}$.

[1] In the margin notes, when we speak of operations, e.g.,'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

2. In our next example, the unknown is t and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$\begin{array}{rcll}
3-1.7t & = & \dfrac{t}{4} & \\
40(3-1.7t) & = & 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3)-40(1.7t) & = & \dfrac{40t}{4} & \text{Distribute} \\
120-68t & = & 10t & \\
(120-68t)+68t & = & 10t+68t & \text{Add } 68t \text{ to both sides} \\
120 & = & 78t & 68t+10t=(68+10)t=78t \\
\dfrac{120}{78} & = & \dfrac{78t}{78} & \text{Divide by the coefficient of } t \\
\dfrac{120}{78} & = & t & \\
\dfrac{20}{13} & = & t & \text{Reduce to lowest terms}
\end{array}$$

To check, we again substitute $t=\frac{20}{13}$ into each side of the original equation. We find that $3-1.7\left(\frac{20}{13}\right)=3-\left(\frac{17}{10}\right)\left(\frac{20}{13}\right)=\frac{5}{13}$ and $\frac{(20/13)}{4}=\frac{20}{13}\cdot\frac{1}{4}=\frac{5}{13}$ as well.

3. To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$\begin{array}{rcll}
\dfrac{1}{18}(7-4a)+2 & = & \dfrac{a}{3}-\dfrac{4-a}{12} & \\
36\left(\dfrac{1}{18}(7-4a)+2\right) & = & 36\left(\dfrac{a}{3}-\dfrac{4-a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7-4a)+(36)(2) & = & \dfrac{36a}{3}-\dfrac{36(4-a)}{12} & \text{Distribute} \\
2(7-4a)+72 & = & 12a-3(4-a) & \text{Distribute} \\
14-8a+72 & = & 12a-12+3a & \\
86-8a & = & 15a-12 & 12a+3a=(12+3)a=15a \\
(86-8a)+8a+12 & = & (15a-12)+8a+12 & \text{Add } 8a \text{ and } 12 \\
86+12-8a+8a & = & 15a+8a-12+12 & \text{Rearrange terms} \\
98 & = & 23a & 15a+8a=(15+8)a=23a \\
\dfrac{98}{23} & = & \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} & = & a &
\end{array}$$

The check, as usual, involves substituting $a=\frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case y) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
(8y\sqrt{3}+1)-1-2y\sqrt{3} & = & (7-10\sqrt{3}+2y\sqrt{3})-1-2y\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
8y\sqrt{3}-2y\sqrt{3}+1-1 & = & 7-1-10\sqrt{3}+2y\sqrt{3}-2y\sqrt{3} & \text{Rearrange terms} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
\dfrac{6y\sqrt{3}}{6\sqrt{3}} & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide } 6\sqrt{3} \\
y & = & \dfrac{2\cdot\sqrt{3}\cdot\sqrt{3}-2\cdot 5\cdot\sqrt{3}}{2\cdot 3\cdot\sqrt{3}} & \\
y & = & \dfrac{\cancel{2}\cancel{\sqrt{3}}(\sqrt{3}-5)}{\cancel{2}\cdot 3\cdot\cancel{\sqrt{3}}} & \text{Factor and cancel} \\
y & = & \dfrac{\sqrt{3}-5}{3} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3}=6-10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3}=6-10\sqrt{3}$ is in fact linear according to Definition 1.10: the variable is y, the value of A is $6\sqrt{3}$ and $B=6-10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing x on one side and move the other terms to the other, we factor out x to

identify its coefficient then divide to get our answer.

$$\begin{array}{rcll}
\frac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\frac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=\sqrt{25\cdot 2} \\
2\left(\frac{3x-1}{2}\right) & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by 2} \\
\frac{\not{2}\cdot(3x-1)}{\not{2}} & = & 2(5x\sqrt{2})+2\cdot 4 & \text{Distribute} \\
3x-1 & = & 10x\sqrt{2}+8 & \\
(3x-1)-10x\sqrt{2}+1 & = & (10x\sqrt{2}+8)-10x\sqrt{2}+1 & \text{Subtract } 10x\sqrt{2}\text{, add 1} \\
3x-10x\sqrt{2}-1+1 & = & 10x\sqrt{2}-10x\sqrt{2}+8+1 & \text{Rearrange terms} \\
3x-10x\sqrt{2} & = & 9 & \\
(3-10\sqrt{2})x & = & 9 & \text{Factor} \\
\frac{(3-10\sqrt{2})x}{3-10\sqrt{2}} & = & \frac{9}{3-10\sqrt{2}} & \text{Divide by the coefficient of } x \\
x & = & \frac{9}{3-10\sqrt{2}} &
\end{array}$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\frac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. If we were instructed to solve our last equation for x, we'd be done in one step: divide both sides by $(4-y)$ - assuming $4-y\neq 0$, that is. Alas, we are instructed to solve for y, which means we have some more work to do.

$$\begin{array}{rcll}
x(4-y) & = & 8y & \\
4x-xy & = & 8y & \text{Distribute} \\
(4x-xy)+xy & = & 8y+xy & \text{Add } xy \\
4x & = & (8+x)y & \text{Factor}
\end{array}$$

In order to finish the problem, we need to divide both sides of the equation by the coefficient of y which in this case is $8+x$. Since this expression contains a variable, we need to stipulate that we may perform this division only if $8+x\neq 0$, or, in other words, $x\neq -8$. Hence, we write our solution as:

$$y=\frac{4x}{8+x}, \quad \text{provided } x\neq -8$$

What happens if $x=-8$? Substituting $x=-8$ into the original equation gives $(-8)(4-y)=8y$ or $-32+8y=8y$. This reduces to $-32=0$, which is a contradiction. This means there is no solution when $x=-8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to show that both sides work out to $\frac{32x}{x+8}$. □

1.3.2 Linear Inequalities

We now turn our attention to linear inequalities. Unlike linear equations which admit at most one solution, the solutions to linear inequalities are generally intervals of real numbers. While the solution strategy for solving linear inequalities is the same as with solving linear equations, we need to remind ourselves that, should we decide to multiply or divide both sides of an inequality by a **negative** number, we need to reverse the direction of the inequality. (See page 38.) In the example below, we work not only some 'simple' linear inequalities in the sense there is only one inequality present, but also some 'compound' linear inequalities which require us to use the notions of intersection and union.

Example 1.3.2. Solve the following inequalities for the indicated variable.

1. Solve for x: $\dfrac{7-8x}{2} \geq 4x+1$
2. Solve for y: $\dfrac{3}{4} \leq \dfrac{7-y}{2} < 6$
3. Solve for t: $2t-1 \leq 4-t < 6t+1$
4. Solve for x: $5+\sqrt{7}x \leq 4x+1 \leq 8$
5. Solve for w: $2.1-0.01w \leq -3$ or $2.1-0.01w \geq 3$

Solution.

1. We begin by clearing denominators and gathering all of the terms containing x to one side of the inequality and putting the remaining terms on the other.

$$\begin{array}{rcll}
\dfrac{7-8x}{2} & \geq & 4x+1 & \\
2\left(\dfrac{7-8x}{2}\right) & \geq & 2(4x+1) & \text{Multiply by 2} \\
\dfrac{\cancel{2}(7-8x)}{\cancel{2}} & \geq & 2(4x)+2(1) & \text{Distribute} \\
7-8x & \geq & 8x+2 & \\
(7-8x)+8x-2 & \geq & 8x+2+8x-2 & \text{Add } 8x\text{, subtract 2} \\
7-2-8x+8x & \geq & 8x+8x+2-2 & \text{Rearrange terms} \\
5 & \geq & 16x & 8x+8x=(8+8)x=16x \\
\dfrac{5}{16} & \geq & \dfrac{16x}{16} & \text{Divide by the coefficient of } x \\
\dfrac{5}{16} & \geq & x &
\end{array}$$

We get $\frac{5}{16} \geq x$ or, said differently, $x \leq \frac{5}{16}$. We express this set[2] of real numbers as $\left(-\infty, \frac{5}{16}\right]$. Though not required to do so, we could partially check our answer by substituting $x = \frac{5}{16}$ and a few other values in our solution set ($x = 0$, for instance) to make sure the inequality holds.

[2]Using set-builder notation, our 'set' of solutions here is $\{x \,|\, x \leq \frac{5}{16}\}$.

(It also isn't a bad idea to choose an $x > \frac{5}{16}$, say $x = 1$, to see that the inequality *doesn't* hold there.) The only real way to actually show that our answer works for *all* values in our solution set is to start with $x \leq \frac{5}{16}$ and reverse all of the steps in our solution procedure to prove it is equivalent to our original inequality.

2. We have our first example of a 'compound' inequality. The solutions to

$$\frac{3}{4} \leq \frac{7-y}{2} < 6$$

must satisfy

$$\frac{3}{4} \leq \frac{7-y}{2} \qquad \underline{\text{and}} \qquad \frac{7-y}{2} < 6$$

One approach is to solve each of these inequalities separately, then intersect their solution sets. While this method works (and will be used later for more complicated problems), since our variable y appears only in the middle expression, we can proceed by essentially working both inequalities at once:

$$\begin{array}{rclll}
\frac{3}{4} \leq & \frac{7-y}{2} & < 6 & & \\
4\left(\frac{3}{4}\right) \leq & 4\left(\frac{7-y}{2}\right) & < 4(6) & & \text{Multiply by 4} \\
\frac{\cancel{4} \cdot 3}{\cancel{4}} \leq & \frac{\overset{2}{\cancel{4}}(7-y)}{\cancel{2}} & < 24 & & \\
3 \leq & 2(7-y) & < 24 & & \\
3 \leq & 2(7) - 2y & < 24 & & \text{Distrbute} \\
3 \leq & 14 - 2y & < 24 & & \\
3 - 14 \leq & (14-2y) - 14 & < 24 - 14 & & \text{Subtract 14} \\
-11 \leq & -2y & < 10 & & \\
\frac{-11}{-2} \geq & \frac{-2y}{-2} & > \frac{10}{-2} & & \text{Divide by the coefficient of } y \\
& & & & \text{Reverse inequalities} \\
\frac{11}{2} \geq & y & > -5 & &
\end{array}$$

Our final answer is $\frac{11}{2} \geq y > -5$, or, said differently, $-5 < y \leq \frac{11}{2}$. In interval notation, this is $\left(-5, \frac{11}{2}\right]$. We could check the reasonableness of our answer as before, and the reader is encouraged to do so.

3. We have another compound inequality and what distinguishes this one from our previous example is that 't' appears on both sides of both inequalities. In this case, we need to create two separate inequalities and find all of the real numbers t which satisfy both $2t - 1 \leq 4 - t$ *and* $4 - t < 6t + 1$. The first inequality, $2t - 1 \leq 4 - t$, reduces to $3t \leq 5$ or $t \leq \frac{5}{3}$. The second inequality, $4 - t < 6t + 1$, becomes $3 < 7t$ which reduces to $t > \frac{3}{7}$. Thus our solution is all

real numbers t with $t \leq \frac{5}{3}$ *and* $t > \frac{3}{7}$, or, writing this as a compound inequality, $\frac{3}{7} < t \leq \frac{5}{3}$. Using interval notation,[3] we express our solution as $\left(\frac{3}{7}, \frac{5}{3}\right]$.

4. As before, with this inequality we have no choice but to solve each inequality individually and intersect the solution sets. Starting with the leftmost inequality, we first note that the in the term $\sqrt{7}x$, the vinculum of the square root extends over the 7 only, meaning the x is not part of the radicand. In order to avoid confusion, we will write $\sqrt{7}x$ as $x\sqrt{7}$.

$$\begin{array}{rcll} 5+x\sqrt{7} & \leq & 4x+1 & \\ (5+x\sqrt{7})-4x-5 & \leq & (4x+1)-4x-5 & \text{Subtract } 4x \text{ and } 5 \\ x\sqrt{7}-4x+5-5 & \leq & 4x-4x+1-5 & \text{Rearrange terms} \\ x(\sqrt{7}-4) & \leq & -4 & \text{Factor} \end{array}$$

At this point, we need to exercise a bit of caution because the number $\sqrt{7}-4$ is negative.[4] When we divide by it the inequality reverses:

$$\begin{array}{rcll} x(\sqrt{7}-4) & \leq & -4 & \\ \dfrac{x(\sqrt{7}-4)}{\sqrt{7}-4} & \geq & \dfrac{-4}{\sqrt{7}-4} & \begin{array}{r}\text{Divide by the coefficient of } x \\ \text{Reverse inequalities}\end{array} \\ x & \geq & \dfrac{-4}{\sqrt{7}-4} & \\ x & \geq & \dfrac{-4}{-(4-\sqrt{7})} & \\ x & \geq & \dfrac{4}{4-\sqrt{7}} & \end{array}$$

We're only half done because we still have the rightmost inequality to solve. Fortunately, that one seems rather mundane: $4x+1 \leq 8$ reduces to $x \leq \frac{7}{4}$ without too much incident. Our solution is $x \geq \frac{4}{4-\sqrt{7}}$ *and* $x \leq \frac{7}{4}$. We may be tempted to write $\frac{4}{4-\sqrt{7}} \leq x \leq \frac{7}{4}$ and call it a day but that would be nonsense! To see why, notice that $\sqrt{7}$ is between 2 and 3 so $\frac{4}{4-\sqrt{7}}$ is between $\frac{4}{4-2} = 2$ and $\frac{4}{4-3} = 4$. In particular, we get $\frac{4}{4-\sqrt{7}} > 2$. On the other hand, $\frac{7}{4} < 2$. This means that our 'solutions' have to be simultaneously greater than 2 AND less than 2 which is impossible. Therefore, this compound inequality has no solution, which means we did all that work for nothing.[5]

5. Our last example is yet another compound inequality but here, instead of the two inequalities being connected with the conjunction '*and*', they are connected with '*or*', which indicates that we need to find the *union* of the results of each. Starting with $2.1 - 0.01w \leq -3$, we get $-0.01w \leq -5.1$, which gives[6] $w \geq 510$. The second inequality, $2.1 - 0.01w \geq 3$, becomes

[3] If we intersect the solution sets of the two individual inequalities, we get the answer, too: $\left(-\infty, \frac{5}{3}\right] \cap \left(\frac{3}{7}, \infty\right) = \left(\frac{3}{7}, \frac{5}{3}\right]$.
[4] Since $4 < 7 < 9$, it stands to reason that $\sqrt{4} < \sqrt{7} < \sqrt{9}$ so $2 < \sqrt{7} < 3$.
[5] Much like how people walking on treadmills get nowhere. Math is the endurance cardio of the brain, folks!
[6] Don't forget to flip the inequality!

$-0.01w \geq 0.9$, which reduces to $w \leq -90$. Our solution set consists of all real numbers w with $w \geq 510$ *or* $w \leq -90$. In interval notation, this is $(-\infty, -90] \cup [510, \infty)$. □

1.3.3 Exercises

In Exercises 1 - 9, solve the given linear equation and check your answer.

1. $3x - 4 = 2 - 4(x - 3)$
2. $\dfrac{3 - 2t}{4} = 7t + 1$
3. $\dfrac{2(w - 3)}{5} = \dfrac{4}{15} - \dfrac{3w + 1}{9}$
4. $-0.02y + 1000 = 0$
5. $\dfrac{49w - 14}{7} = 3w - (2 - 4w)$
6. $7 - (4 - x) = \dfrac{2x - 3}{2}$
7. $3t\sqrt{7} + 5 = 0$
8. $\sqrt{50}y = \dfrac{6 - \sqrt{8}y}{3}$
9. $4 - (2x + 1) = \dfrac{x\sqrt{7}}{9}$

In equations 10 - 27, solve each equation for the indicated variable.

10. Solve for y: $3x + 2y = 4$
11. Solve for x: $3x + 2y = 4$
12. Solve for C: $F = \dfrac{9}{5}C + 32$
13. Solve for x: $p = -2.5x + 15$
14. Solve for x: $C = 200x + 1000$
15. Solve for y: $x = 4(y + 1) + 3$
16. Solve for w: $vw - 1 = 3v$
17. Solve for v: $vw - 1 = 3v$
18. Solve for y: $x(y - 3) = 2y + 1$
19. Solve for π: $C = 2\pi r$
20. Solve for V: $PV = nRT$
21. Solve for R: $PV = nRT$
22. Solve for g: $E = mgh$
23. Solve for m: $E = \dfrac{1}{2}mv^2$

In Exercises 24 - 27, the subscripts on the variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat 'P_1' and 'P_2' as two different variables as you would 'x' and 'y.' (The same goes for 'x' and 'x_0,' etc.)

24. Solve for V_2: $P_1V_1 = P_2V_2$
25. Solve for t: $x = x_0 + at$
26. Solve for x: $y - y_0 = m(x - x_0)$
27. Solve for T_1: $q = mc(T_2 - T_1)$
28. With the help of your classmates, find values for c so that the equation: $2x - 5c = 1 - c(x + 2)$

    (a) has $x = 42$ as a solution.
    (b) has no solution (that is, the equation is a contradiction.)

    Is it possible to find a value of c so the equation is an identity? Explain.

In Exercises 29 - 46, solve the given inequality. Write your answer using interval notation.

29. $3-4x \geq 0$

30. $2t-1 < 3-(4t-3)$

31. $\dfrac{7-y}{4} \geq 3y+1$

32. $0.05R+1.2 > 0.8-0.25R$

33. $7-(2-x) \leq x+3$

34. $\dfrac{10m+1}{5} \geq 2m-\dfrac{1}{2}$

35. $x\sqrt{12}-\sqrt{3} > \sqrt{3}x+\sqrt{27}$

36. $2t-7 \leq \sqrt[3]{18}t$

37. $117y \geq y\sqrt{2}-7y\sqrt[4]{8}$

38. $-\dfrac{1}{2} \leq 5x-3 \leq \dfrac{1}{2}$

39. $-\dfrac{3}{2} \leq \dfrac{4-2t}{10} < \dfrac{7}{6}$

40. $-0.1 \leq \dfrac{5-x}{3}-2 < 0.1$

41. $2y \leq 3-y < 7$

42. $3x \geq 4-x \geq 3$

43. $6-5t > \dfrac{4t}{3} \geq t-2$

44. $2x+1 \leq -1$ or $2x+1 \geq 1$

45. $4-x \leq 0$ or $2x+7 < x$

46. $\dfrac{5-2x}{3} > x$ or $2x+5 \geq 1$

1.3.4 Answers

1. $x = \frac{18}{7}$
2. $t = -\frac{1}{30}$
3. $w = \frac{61}{33}$
4. $y = 50000$
5. All real numbers.
6. No solution.
7. $t = -\frac{5}{3\sqrt{7}} = -\frac{5\sqrt{7}}{21}$
8. $y = \frac{6}{17\sqrt{2}} = \frac{3\sqrt{2}}{17}$
9. $x = \frac{27}{18+\sqrt{7}}$
10. $y = \frac{4-3x}{2}$ or $y = -\frac{3}{2}x + 2$
11. $x = \frac{4-2y}{3}$ or $x = -\frac{2}{3}y + \frac{4}{3}$
12. $C = \frac{5}{9}(F - 32)$ or $C = \frac{5}{9}F - \frac{160}{9}$
13. $x = \frac{p-15}{-2.5} = \frac{15-p}{2.5}$ or $x = -\frac{2}{5}p + 6$.
14. $x = \frac{C-1000}{200}$ or $x = \frac{1}{200}C - 5$
15. $y = \frac{x-7}{4}$ or $y = \frac{1}{4}x - \frac{7}{4}$
16. $w = \frac{3v+1}{v}$, provided $v \neq 0$.
17. $v = \frac{1}{w-3}$, provided $w \neq 3$.
18. $y = \frac{3x+1}{x-2}$, provided $x \neq 2$.
19. $\pi = \frac{C}{2r}$, provided $r \neq 0$.
20. $V = \frac{nRT}{P}$, provided $P \neq 0$.
21. $R = \frac{PV}{nT}$, provided $n \neq 0$, $T \neq 0$.
22. $g = \frac{E}{mh}$, provided $m \neq 0$, $h \neq 0$.
23. $m = \frac{2E}{v^2}$, provided $v^2 \neq 0$ (so $v \neq 0$).
24. $V_2 = \frac{P_1 V_1}{P_2}$, provided $P_2 \neq 0$.
25. $t = \frac{x - x_0}{a}$, provided $a \neq 0$.
26. $x = \frac{y - y_0 + mx_0}{m}$ or $x = x_0 + \frac{y - y_0}{m}$, provided $m \neq 0$.
27. $T_1 = \frac{mcT_2 - q}{mc}$ or $T_1 = T_2 - \frac{q}{mc}$, provided $m \neq 0$, $c \neq 0$.
29. $\left(-\infty, \frac{3}{4}\right]$
30. $\left(-\infty, \frac{7}{6}\right)$
31. $\left(-\infty, \frac{3}{13}\right]$
32. $\left(-\frac{4}{3}, \infty\right)$
33. No solution.
34. $(-\infty, \infty)$
35. $(4, \infty)$
36. $\left[\frac{7}{2 - \sqrt[3]{18}}, \infty\right)$
37. $[0, \infty)$

38. $\left[\frac{1}{2}, \frac{7}{10}\right]$

39. $\left(-\frac{23}{6}, \frac{19}{2}\right]$

40. $\left(-\frac{13}{10}, -\frac{7}{10}\right]$

41. $(-4, 1]$

42. $\{1\} = [1, 1]$

43. $\left[-6, \frac{18}{19}\right)$

44. $(-\infty, -1] \cup [0, \infty)$

45. $(-\infty, -7) \cup [4, \infty)$

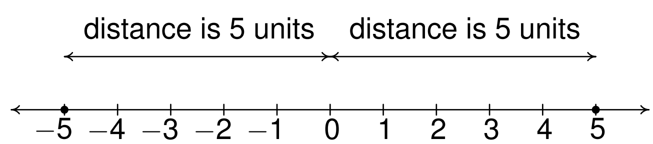
46. $(-\infty, \infty)$

1.4 Absolute Value Equations and Inequalities

In this section, we review some basic concepts involving the absolute value of a real number x. There are a few different ways to define absolute value and in this section we choose the following definition. (Absolute value will be revisited in much greater depth in Section **??** where we present what one can think of as the "precise" definition.)

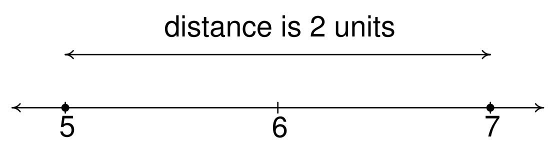
> **Definition 1.11. Absolute Value as Distance:** For every real number x, the **absolute value** of x, denoted $|x|$, is the distance between x and 0 on the number line. More generally, if x and c are real numbers, $|x - c|$ is the distance between the numbers x and c on the number line.

For example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line:



Graphically why $|-5| = 5$ and $|5| = 5$

Computationally, the absolute value 'makes negative numbers positive', though we need to be a little cautious with this description. While $|-7| = 7$, $|5 - 7| \neq 5 + 7$. The absolute value acts as a grouping symbol, so $|5 - 7| = |-2| = 2$, which makes sense since 5 and 7 are two units away from each other on the number line:



Graphically why $|5 - 7| = 2$

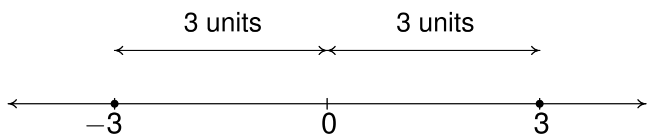
We list some of the operational properties of absolute value below.

> **Theorem 1.2. Properties of Absolute Value:** Let a, b and x be real numbers and let n be an integer.[a] Then
>
> - **Product Rule:** $|ab| = |a||b|$
> - **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
> - **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$
>
> [a]See page 6 if you don't remember what an integer is.

The proof of Theorem 1.2 is difficult, but not impossible, using the distance definition of absolute value or even the 'it makes negatives positive' notion. It is, however, much easier if one uses the "precise" definition given in Section **??** so we will revisit the proof then. For now, let's focus on how to solve basic equations and inequalities involving the absolute value.

1.4.1 Absolute Value Equations

Thinking of absolute value in terms of distance gives us a geometric way to interpret equations. For example, to solve $|x| = 3$, we are looking for all real numbers x whose distance from 0 is 3 units. If we move three units to the right of 0, we end up at $x = 3$. If we move three units to the left, we end up at $x = -3$. Thus the solutions to $|x| = 3$ are $x = \pm 3$.



The solutions to $|x| = 3$ are $x = \pm 3$.

Thinking this way gives us the following.

Theorem 1.3. Absolute Value Equations: Suppose x, y and c are real numbers.

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.
- $|x| = |y|$ if and only if $x = y$ or $x = -y$.
  (That is, if two numbers have the same absolute values, they are either the same number or exact opposites.)

Theorem 1.3 is our main tool in solving equations involving the absolute value, since it allows us a way to rewrite such equations as compound linear equations.

Strategy for Solving Equations Involving Absolute Value

In order to solve an equation involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the equation so it has the form $|X| = c$.
2. Apply Theorem 1.3.

The techniques we use to 'isolate the absolute value' are precisely those we used in Section 1.3 to isolate the variable when solving linear equations. Time for some practice.

Example 1.4.1. Solve each of the following equations.

1. $|3x - 1| = 6$
2. $\dfrac{3 - |y + 5|}{2} = 1$
3. $3|2t + 1| - \sqrt{5} = 0$

4. $4 - |5w + 3| = 5$
5. $\left|3 - x\sqrt[3]{12}\right| = |4x + 1|$
6. $|t - 1| - 3|t + 1| = 0$

Solution.

1. The equation $|3x - 1| = 6$ is of already in the form $|X| = c$, so we know $3x - 1 = 6$ or $3x - 1 = -6$. Solving the former gives us at $x = \frac{7}{3}$ and solving the latter yields $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. We begin solving $\frac{3-|y+5|}{2} = 1$ by isolating the absolute value to put it in the form $|X| = c$.

$$\begin{array}{rcll} \dfrac{3 - |y+5|}{2} & = & 1 & \\ 3 - |y+5| & = & 2 & \text{Multiply by } 2 \\ -|y+5| & = & -1 & \text{Subtract } 3 \\ |y+5| & = & 1 & \text{Divide by } -1 \end{array}$$

At this point, we have $y + 5 = 1$ or $y + 5 = -1$, so our solutions are $y = -4$ or $y = -6$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value. Don't let the $\sqrt{5}$ throw you off - it's just another real number, so we treat it as such:

$$\begin{array}{rcll} 3|2t+1| - \sqrt{5} & = & 0 & \\ 3|2t+1| & = & \sqrt{5} & \text{Add } \sqrt{5} \\ |2t+1| & = & \dfrac{\sqrt{5}}{3} & \text{Divide by } 3 \end{array}$$

From here, we have that $2t + 1 = \frac{\sqrt{5}}{3}$ or $2t + 1 = -\frac{\sqrt{5}}{3}$. The first equation gives $t = \frac{\sqrt{5}-3}{6}$ while the second gives $t = \frac{-\sqrt{5}-3}{6}$ thus we list our answers as $t = \frac{-3\pm\sqrt{5}}{6}$. The reader should enjoy the challenge of substituting both answers into the original equation and following through the arithmetic to see that both answers work.

4. Upon isolating the absolute value in the equation $4 - |5w + 3| = 5$, we get $|5w + 3| = -1$. At this point, we know there cannot be any real solution. By definition, the absolute value is a *distance*, and as such is never negative. We write 'no solution' and carry on.

5. Our next equation already has the absolute value expressions (plural) isolated, so we work from the principle that if $|x| = |y|$, then $x = y$ or $x = -y$. Thus from $\left|3 - x\sqrt[3]{12}\right| = |4x + 1|$ we get two equations to solve:

$$3 - x\sqrt[3]{12} = 4x + 1, \qquad \text{and} \qquad 3 - x\sqrt[3]{12} = -(4x + 1)$$

Notice that the right side of the second equation is $-(4x+1)$ and not simply $-4x+1$. Remember, the expression $4x + 1$ represents a single real number so in order to negate it we need to negate the *entire* expression $-(4x+1)$. Moving along, when solving $3 - x\sqrt[3]{12} = 4x + 1$, we obtain $x = \frac{2}{4+\sqrt[3]{12}}$ and the solution to $3 - x\sqrt[3]{12} = -(4x+1)$ is $x = \frac{4}{\sqrt[3]{12}-4}$. As usual, the reader is invited to check these answers by substituting them into the original equation.

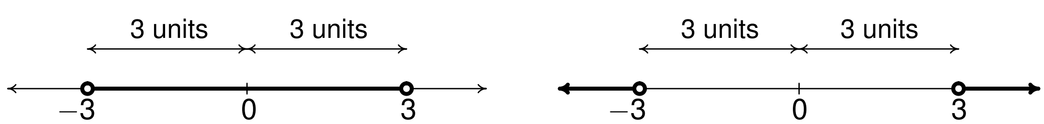
6. We start by isolating one of the absolute value expressions: $|t-1| - 3|t+1| = 0$ gives $|t-1| = 3|t+1|$. While this *resembles* the form $|x| = |y|$, the coefficient 3 in $3|t+1|$ prevents it from being an exact match. Not to worry - since 3 is positive, $3 = |3|$ so

$$3|t+1| = |3||t+1| = |3(t+1)| = |3t+3|.$$

Hence, our equation becomes $|t-1| = |3t+3|$ which results in the two equations: $t-1 = 3t+3$ and $t-1 = -(3t+3)$. The first equation gives $t = -2$ and the second gives $t = -\frac{1}{2}$. The reader is encouraged to check both answers in the original equation. □

1.4.2 Absolute Value Inequalities

We now turn our attention to solving some basic inequalities involving the absolute value. Suppose we wished to solve $|x| < 3$. Geometrically, we are looking for all of the real numbers whose distance from 0 is *less* than 3 units. We get $-3 < x < 3$, or in interval notation, $(-3, 3)$. Suppose we are asked to solve $|x| > 3$ instead. Now we want the distance between x and 0 to be *greater* than 3 units. Moving in the positive direction, this means $x > 3$. In the negative direction, this puts $x < -3$. Our solutions would then satisfy $x < -3$ *or* $x > 3$. In interval notation, we express this as $(-\infty, -3) \cup (3, \infty)$.



The solution to $|x| < 3$ is $(-3, 3)$ The solution to $|x| > 3$ is $(-\infty, -3) \cup (3, \infty)$

Generalizing this notion, we get the following:

Theorem 1.4. Inequalities Involving Absolute Value: Let c be a real number.

- If $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- If $c \leq 0$, $|x| < c$ has no solution.
- If $c > 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- If $c \leq 0$, $|x| > c$ is true for all real numbers.

If the inequality we're faced with involves '$\leq$' or '$\geq$,' we can combine the results of Theorem 1.4 with Theorem 1.3 as needed.

Strategy for Solving Inequalities Involving Absolute Value

In order to solve an inequality involving the absolute value of a quantity $|X|$:

1. Isolate the absolute value on one side of the inequality.
2. Apply Theorem 1.4.

Example 1.4.2. Solve the following inequalities.

1. $\left|x - \sqrt[4]{5}\right| > 1$
2. $\dfrac{4 - 2|2x+1|}{4} \geq -\sqrt{3}$
3. $|2x - 1| \leq 3|4 - 8x| - 10$
4. $|2x - 1| \leq 3|4 - 8x| + 10$
5. $2 < |x - 1| \leq 5$
6. $|10x - 5| + |10 - 5x| \leq 0$

Solution.

1. From Theorem 1.4, $\left|x - \sqrt[4]{5}\right| > 1$ is equivalent to $x - \sqrt[4]{5} < -1$ or $x - \sqrt[4]{5} > 1$. Solving this compound inequality, we get $x < -1 + \sqrt[4]{5}$ or $x > 1 + \sqrt[4]{5}$. Our answer, in interval notation, is: $\left(-\infty, -1 + \sqrt[4]{5}\right) \cup \left(1 + \sqrt[4]{5}, \infty\right)$. As with linear inequalities, we can partially check our answer by selecting values of x both inside and outside the solution intervals to see which values of x satisfy the original inequality and which do not.

2. Our first step in solving $\frac{4-2|2x+1|}{4} \geq -\sqrt{3}$ is to isolate the absolute value.

$$\begin{array}{rcll}
\dfrac{4 - 2|2x+1|}{4} & \geq & -\sqrt{3} & \\
4 - 2|2x+1| & \geq & -4\sqrt{3} & \text{Multiply by 4} \\
-2|2x+1| & \geq & -4 - 4\sqrt{3} & \text{Subtract 4} \\
|2x+1| & \leq & \dfrac{-4 - 4\sqrt{3}}{-2} & \text{Divide by } -2\text{, reverse the inequality} \\
|2x+1| & \leq & 2 + 2\sqrt{3} & \text{Reduce}
\end{array}$$

Since we're dealing with '$\leq$' instead of just '$<$,' we can combine Theorems 1.4 and 1.3 to rewrite this last inequality as:[1] $-(2 + 2\sqrt{3}) \leq 2x + 1 \leq 2 + 2\sqrt{3}$. Subtracting the '1' across both inequalities gives $-3 - 2\sqrt{3} \leq 2x \leq 1 + 2\sqrt{3}$, which reduces to $\frac{-3-2\sqrt{3}}{2} \leq x \leq \frac{1+2\sqrt{3}}{2}$. In interval notation this reads as $\left[\frac{-3-2\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right]$.

[1]Note the use of parentheses: $-(2 + 2\sqrt{3})$ as opposed to $-2 + 2\sqrt{3}$.

3. There are two absolute values in $|2x-1| \leq 3|4-8x|-10$, so it is unclear how we are to proceed. However, before jumping in and trying to apply (or misapply) Theorem 1.4, we note that $|4-8x| = |(-4)(2x-1)|$. Using this, we get:

$$\begin{array}{rcll} |2x-1| & \leq & 3|4-8x|-10 & \\ |2x-1| & \leq & 3|(-4)(2x-1)|-10 & \text{Factor} \\ |2x-1| & \leq & 3|-4||2x-1|-10 & \text{Product Rule} \\ |2x-1| & \leq & 12|2x-1|-10 & \\ -11|2x-1| & \leq & -10 & \text{Subtract } 12|2x-1| \\ |2x-1| & \geq & \dfrac{10}{11} & \text{Divide by } -11 \text{ and reduce} \end{array}$$

At this point, we invoke Theorems 1.3 and 1.4 and write the equivalent compound inequality: $2x-1 \leq -\frac{10}{11}$ *or* $2x-1 \geq \frac{10}{11}$. We get $x \leq \frac{1}{22}$ *or* $x \geq \frac{21}{22}$, which, in interval notation reads $\left(-\infty, \frac{1}{22}\right] \cup \left[\frac{21}{22}, \infty\right)$.

4. The inequality $|2x-1| \leq 3|4-8x|+10$ differs from the previous example in exactly one respect: on the right side of the inequality, we have '+10' instead of '−10.' The steps to isolate the absolute value here are identical to those in the previous example, but instead of obtaining $|2x-1| \geq \frac{10}{11}$ as before, we obtain $|2x-1| \geq -\frac{10}{11}$. This latter inequality is *always* true. (Absolute value is, by definition, a distance and hence always 0 or greater.) Thus our solution to this inequality is all real numbers, $(-\infty, \infty)$.

5. To solve $2 < |x-1| \leq 5$, we rewrite it as the compound inequality: $2 < |x-1|$ *and* $|x-1| \leq 5$. The first inequality, $2 < |x-1|$, can be re-written as $|x-1| > 2$ so it is equivalent to $x-1 < -2$ *or* $x-1 > 2$. Thus the solution to $2 < |x-1|$ is $x < -1$ or $x > 3$, which in interval notation is $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \leq 5$, we combine the results of Theorems 1.3 and 1.4 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we intersect $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to get our final answer $[-4, -1) \cup (3, 6]$.

6. Our first hope when encountering $|10x-5| + |10-5x| \leq 0$ is that we can somehow combine the two absolute value quantities as we'd done in earlier examples. We leave it to the reader to show, however, that no matter what we try to factor out of the absolute value quantities, what remains inside the absolute values will always be different. At this point, we take a step back and look at the equation in a more general way: we are adding two absolute values together and wanting the result to be less than or equal to 0. Since the absolute value of anything is always 0 or greater, there are no solutions to: $|10x-5| + |10-5x| < 0$. Is it possible that $|10x-5| + |10-5x| = 0$? Only if there is an x where $|10x-5| = 0$ and $|10-5x| = 0$ *at the same time*.[2] The first equation holds only when $x = \frac{1}{2}$, while the second holds only when $x = 2$. Alas, we have no solution.[3] □

[2]Do you see why?
[3]Not for lack of trying, however!

We close this section with an example of how the properties in Theorem 1.2 are used in Calculus. Here, 'ε' is the Greek letter 'epsilon' and it represents a positive real number. Those of you who will be taking Calculus in the future should become *very* familiar with this type of algebraic manipulation.

$$\begin{aligned}
\left|\frac{8-4x}{3}\right| &< \varepsilon \\
\frac{|8-4x|}{|3|} &< \varepsilon && \text{Quotient Rule} \\
\frac{|-4(x-2)|}{3} &< \varepsilon && \text{Factor} \\
\frac{|-4||x-2|}{3} &< \varepsilon && \text{Product Rule} \\
\frac{4|x-2|}{3} &< \varepsilon \\
\frac{3}{4}\cdot\frac{4|x-2|}{3} &< \frac{3}{4}\cdot\varepsilon && \text{Multiply by } \frac{3}{4} \\
|x-2| &< \frac{3}{4}\varepsilon
\end{aligned}$$

1.4.3 Exercises

In Exercises 1 - 18, solve the equation.

1. $|x| = 6$
2. $|3t - 1| = 10$
3. $|4 - w| = 7$
4. $4 - |y| = 3$
5. $2|5m + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\dfrac{2}{3}|5 - 2w| - \dfrac{1}{2} = 5$
9. $|3t - \sqrt{2}| + 4 = 6$
10. $\dfrac{|2v + 1| - 3}{4} = \dfrac{1}{2} - |2v+1|$
11. $|2x + 1| = \dfrac{|2x + 1| - 3}{2}$
12. $\dfrac{|3 - 2y| + 4}{2} = 2 - |3 - 2y|$
13. $|3t - 2| = |2t + 7|$
14. $|3x + 1| = |4x|$
15. $|1 - \sqrt{2}y| = |y + 1|$
16. $|4 - x| - |x + 2| = 0$
17. $|2 - 5z| = 5|z + 1|$
18. $\sqrt{3}|w - 1| = 2|w + 1|$

In Exercises 19 - 30, solve the inequality. Write your answer using interval notation.

19. $|3x - 5| \leq 4$
20. $|7t + 2| > 10$
21. $|2w + 1| - 5 < 0$
22. $|2 - y| - 4 \geq -3$
23. $|3z + 5| + 2 < 1$
24. $2|7 - v| + 4 > 1$
25. $3 - |x + \sqrt{5}| < -3$
26. $|5t| \leq |t| + 3$
27. $|w - 3| < |3 - w|$
28. $2 \leq |4 - y| < 7$
29. $1 < |2w - 9| \leq 3$
30. $3 > 2|\sqrt{3} - x| > 1$
31. With help from your classmates, solve:
    (a) $|5 - |2x - 3|| = 4$
    (b) $|5 - |2x - 3|| < 4$

1.4.4 Answers

1. $x = -6$ or $x = 6$
2. $t = -3$ or $t = \frac{11}{3}$
3. $w = -3$ or $w = 11$
4. $y = -1$ or $y = 1$
5. $m = -\frac{1}{2}$ or $m = \frac{1}{10}$
6. No solution
7. $x = -3$ or $x = 3$
8. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
9. $t = \frac{\sqrt{2} \pm 2}{3}$
10. $v = -1$ or $v = 0$
11. No solution
12. $y = \frac{3}{2}$
13. $t = -1$ or $t = 9$
14. $x = -\frac{1}{7}$ or $x = 1$
15. $y = 0$ or $y = \frac{2}{\sqrt{2} - 1}$
16. $x = 1$
17. $z = -\frac{3}{10}$
18. $w = \frac{\sqrt{3} \pm 2}{\sqrt{3} \mp 2}$ See footnote[4]
19. $\left[\frac{1}{3}, 3\right]$
20. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
21. $(-3, 2)$
22. $(-\infty, 1] \cup [3, \infty)$
23. No solution
24. $(-\infty, \infty)$
25. $(-\infty, -6 - \sqrt{5}) \cup (6 - \sqrt{5}, \infty)$
26. $\left[-\frac{3}{4}, \frac{3}{4}\right]$
27. No solution
28. $(-3, 2] \cup [6, 11)$
29. $[3, 4) \cup (5, 6]$
30. $\left(\frac{2\sqrt{3} - 3}{2}, \frac{2\sqrt{3} - 1}{2}\right) \cup \left(\frac{2\sqrt{3} + 1}{2}, \frac{2\sqrt{3} + 3}{2}\right)$
31. (a) $x = -3$, or $x = 1$, or $x = 2$, or $x = 6$
    (b) $(-3, 1) \cup (2, 6)$

[4]That is, $w = \frac{\sqrt{3} + 2}{\sqrt{3} - 2}$ or $w = \frac{\sqrt{3} - 2}{\sqrt{3} + 2}$

1.5 Polynomial Arithmetic

In this section, we review the arithmetic of **polynomials.** What precisely is a polynomial?

Definition 1.12. A **polynomial** is a sum of terms each of which is a real number or a real number multiplied by one or more variables to natural number powers.

Some examples of polynomials are $x^2 + x\sqrt{3} + 4$, $27x^2y + \frac{7x}{2}$ and 6. Things like $3\sqrt{x}$, $4x - \frac{2}{x+1}$ and $13x^{2/3}y^2$ are **not** polynomials. (Do you see why not?) Below we review some of the terminology associated with polynomials.

Definition 1.13. Polynomial Vocabulary

- **Constant Terms:** Terms in polynomials without variables are called **constant** terms.
- **Coefficient:** In non-constant terms, the real number factor in the expression is called the **coefficient** of the term.
- **Degree:** The **degree** of a non-constant term is the sum of the exponents on the variables in the term; non-zero constant terms are defined to have degree 0. The degree of a polynomial is the highest degree of the nonzero terms.
- **Like Terms:** Terms in a polynomial are called **like** terms if they have the same variables each with the same corresponding exponents.
- **Simplified:** A polynomial is said to be **simplified** if all arithmetic operations have been completed and there are no longer any like terms.
- **Classification by Number of Terms:** A simplified polynomial is called a
  - **monomial** if it has exactly one nonzero term
  - **binomial** if it has exactly two nonzero terms
  - **trinomial** if it has exactly three nonzero terms

For example, $x^2 + x\sqrt{3} + 4$ is a trinomial of degree 2. The coefficient of x^2 is 1 and the constant term is 4. The polynomial $27x^2y + \frac{7x}{2}$ is a binomial of degree 3 ($x^2y = x^2y^1$) with constant term 0.

The concept of 'like' terms really amounts to finding terms which can be combined using the Distributive Property. For example, in the polynomial $17x^2y - 3xy^2 + 7xy^2$, $-3xy^2$ and $7xy^2$ are like terms, since they have the same variables with the same corresponding exponents. This allows us to combine these two terms as follows:

$$17x^2y - 3xy^2 + 7xy^2 = 17x^2y + (-3)xy^2 + 7xy^2 + 17x^2y + (-3 + 7)xy^2 = 17x^2y + 4xy^2$$

Note that even though $17x^2y$ and $4xy^2$ have the same variables, they are not like terms since in the first term we have x^2 and $y = y^1$ but in the second we have $x = x^1$ and $y = y^2$ so the corresponding exponents aren't the same. Hence, $17x^2y + 4xy^2$ is the simplified form of the polynomial.

There are four basic operations we can perform with polynomials: addition, subtraction, multiplication and division. The first three of these operations follow directly from properties of real number arithmetic and will be discussed together first. Division, on the other hand, is a bit more complicated and will be discussed separately.

1.5.1 Polynomial Addition, Subtraction and Multiplication.

Adding and subtracting polynomials comes down to identifying like terms and then adding or subtracting the coefficients of those like terms. Multiplying polynomials comes to us courtesy of the Generalized Distributive Property.

Theorem 1.5. Generalized Distributive Property: To multiply a quantity of n terms by a quantity of m terms, multiply each of the n terms of the first quantity by each of the m terms in the second quantity and add the resulting $n \cdot m$ terms together.

In particular, Theorem 1.5 says that, before combining like terms, a product of an n-term polynomial and an m-term polynomial will generate $(n \cdot m)$-terms. For example, a binomial times a trinomial will produce six terms some of which may be like terms. Thus the simplified end result may have fewer than six terms but you will start with six terms.

A special case of Theorem 1.5 is the famous **F.O.I.L.**, listed here:[1]

Theorem 1.6. F.O.I.L: The terms generated from the product of two binomials: $(a + b)(c + d)$ can be verbalized as follows "Take the sum of:

- the product of the **F**irst terms a and c, ac
- the product of the **O**uter terms a and d, ad
- the product of the **I**nner terms b and c, bc
- the product of the **L**ast terms b and d, bd."

That is, $(a + b)(c + d) = ac + ad + bc + bd$.

Theorem 1.5 is best proved using the technique known as Mathematical Induction which is covered in Section **??**. The result is really nothing more than repeated applications of the Distributive Property so it seems reasonable and we'll use it without proof for now. The other major piece of polynomial multiplication is one of the Power Rules of Exponents from page 26 in Section 1.2, namely $a^n a^m = a^{n+m}$. The Commutative and Associative Properties of addition and multiplication are also used extensively. We put all of these properties to good use in the next example.

[1] We caved to peer pressure on this one. Apparently all of the cool Precalculus books have FOIL in them even though it's redundant once you know how to distribute multiplication across addition. In general, we don't like mechanical shortcuts that interfere with a student's understanding of the material and FOIL is one of the worst.

Example 1.5.1. Perform the indicated operations and simplify.

1. $\left(3x^2 - 2x + 1\right) - (7x - 3)$
2. $4xz^2 - 3z(xz - x + 4)$
3. $(2t + 1)(3t - 7)$
4. $\left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right)$
5. $\left(4w - \frac{1}{2}\right)^2$
6. $\left[2(x + h) - (x + h)^2\right] - \left(2x - x^2\right)$

Solution.

1. We begin 'distributing the negative' as indicated on page 20 in Section 1.2, then we rearrange and combine like term:

$$\begin{array}{rcll} \left(3x^2 - 2x + 1\right) - (7x - 3) & = & 3x^2 - 2x + 1 - 7x + 3 & \text{Distribute} \\ & = & 3x^2 - 2x - 7x + 1 + 3 & \text{Rearrange terms} \\ & = & 3x^2 - 9x + 4 & \text{Combine like terms} \end{array}$$

   Our answer is $3x^2 - 9x + 4$.

2. Following in our footsteps from the previous example, we first distribute the $-3z$ through, then rearrange and combine like terms.

$$\begin{array}{rcll} 4xz^2 - 3z(xz - x + 4) & = & 4xz^2 - 3z(xz) + 3z(x) - 3z(4) & \text{Distribute} \\ & = & 4xz^2 - 3xz^2 + 3xz - 12z & \text{Multiply} \\ & = & xz^2 + 3xz - 12z & \text{Combine like terms} \end{array}$$

   We get our final answer: $xz^2 + 3xz - 12z$

3. At last, we have a chance to use our F.O.I.L. technique:

$$\begin{array}{rcll} (2t + 1)(3t - 7) & = & (2t)(3t) + (2t)(-7) + (1)(3t) + (1)(-7) & \text{F.O.I.L.} \\ & = & 6t^2 - 14t + 3t - 7 & \text{Multiply} \\ & = & 6t^2 - 11t - 7 & \text{Combine like terms} \end{array}$$

   We get $6t^2 - 11t - 7$ as our final answer.

4. We use the Generalized Distributive Property here, multiplying each term in the second quantity first by $3y$, then by $-\sqrt[3]{2}$:

$$\begin{array}{rcl} \left(3y - \sqrt[3]{2}\right)\left(9y^2 + 3\sqrt[3]{2}y + \sqrt[3]{4}\right) & = & 3y\left(9y^2\right) + 3y\left(3\sqrt[3]{2}y\right) + 3y\left(\sqrt[3]{4}\right) \\ & & -\sqrt[3]{2}\left(9y^2\right) - \sqrt[3]{2}\left(3\sqrt[3]{2}y\right) - \sqrt[3]{2}\left(\sqrt[3]{4}\right) \\ & = & 27y^3 + 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 9y^2\sqrt[3]{2} - 3y\sqrt[3]{4} - \sqrt[3]{8} \\ & = & 27y^3 + 9y^2\sqrt[3]{2} - 9y^2\sqrt[3]{2} + 3y\sqrt[3]{4} - 3y\sqrt[3]{4} - 2 \\ & = & 27y^3 - 2 \end{array}$$

   To our surprise and delight, this product reduces to $27y^3 - 2$.

5. Since exponents do **not** distribute across powers,[2] $\left(4w - \frac{1}{2}\right)^2 \neq (4w)^2 - \left(\frac{1}{2}\right)^2$. (We know you knew that.) Instead, we proceed as follows:

$$\begin{aligned}
\left(4w - \frac{1}{2}\right)^2 &= \left(4w - \frac{1}{2}\right)\left(4w - \frac{1}{2}\right) & \\
&= (4w)(4w) + (4w)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)(4w) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) & \text{F.O.I.L.} \\
&= 16w^2 - 2w - 2w + \frac{1}{4} & \text{Multiply} \\
&= 16w^2 - 4w + \frac{1}{4} & \text{Combine like terms}
\end{aligned}$$

Our (correct) final answer is $16w^2 - 4w + \frac{1}{4}$.

6. Our last example has two levels of grouping symbols. We begin simplifying the quantity inside the brackets, squaring out the binomial $(x + h)^2$ in the same way we expanded the square in our last example:

$$(x + h)^2 = (x + h)(x + h) = (x)(x) + (x)(h) + (h)(x) + (h)(h) = x^2 + 2xh + h^2$$

When we substitute this into our expression, we envelope it in parentheses, as usual, so we don't forget to distribute the negative.

$$\begin{aligned}
\left[2(x + h) - (x + h)^2\right] - \left(2x - x^2\right) &= \left[2(x + h) - \left(x^2 + 2xh + h^2\right)\right] - \left(2x - x^2\right) & \text{Substitute} \\
&= \left[2x + 2h - x^2 - 2xh - h^2\right] - \left(2x - x^2\right) & \text{Distribute} \\
&= 2x + 2h - x^2 - 2xh - h^2 - 2x + x^2 & \text{Distribute} \\
&= 2x - 2x + 2h - x^2 + x^2 - 2xh - h^2 & \text{Rearrange terms} \\
&= 2h - 2xh - h^2 & \text{Combine like terms}
\end{aligned}$$

We find no like terms in $2h - 2xh - h^2$ so we are finished. □

We conclude our discussion of polynomial multiplication by showcasing two special products which happen often enough they should be committed to memory.

Theorem 1.7. Special Products: Let a and b be real numbers:

- **Perfect Square:** $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$
- **Difference of Two Squares:** $(a - b)(a + b) = a^2 - b^2$

The formulas in Theorem 1.7 can be verified by working through the multiplication.[3]

[2]See the remarks following the Properties of Exponents on 26.
[3]These are both special cases of F.O.I.L.

1.5.2 Polynomial Long Division.

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$\begin{array}{r|l} & 32 \\ \cline{2-2} 79 & 2585 \\ & -237\downarrow \\ \cline{2-2} & 215 \\ & -158 \\ \cline{2-2} & 57 \end{array}$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32) + 57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$\begin{array}{r|r} & 3\cdot 10 + 2 \\ \cline{2-2} 7\cdot 10 + 9 & 2\cdot 10^3 + 5\cdot 10^2 + 8\cdot 10 + 5 \\ & -\left(2\cdot 10^3 + 3\cdot 10^2 + 7\cdot 10\right) \downarrow \\ \cline{2-2} & 2\cdot 10^2 + 1\cdot 10 + 5 \\ & -\left(1\cdot 10^2 + 5\cdot 10 + 8\right) \\ \cline{2-2} & 5\cdot 10 + 7 \end{array}$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier.[4] Before we dive into examples, we first state a theorem telling us when we can divide two polynomials, and what to expect when we do so.

Theorem 1.8. Polynomial Division: Let d and p be nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, q and r, such that $p = d \cdot q + r$, where either $r = 0$ or the degree of r is strictly less than the degree of d.

Essentially, Theorem 1.8 tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples to refresh your memory.

Example 1.5.2. Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$
2. $(2t + 7) \div (3t - 4)$
3. $\left(6y^2 - 1\right) \div (2y + 5)$
4. $\left(w^3\right) \div \left(w^2 - \sqrt{2}\right)$.

Solution.

1. To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x - 2$, by this first term in the quotient to get $x^2(x - 2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3 + 4x^2 - 5x - 14 \\ & -\underline{\left(x^3 - 2x^2\right)} \quad \downarrow \\ & \qquad 6x^2 - 5x \end{array}$$

Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x - 2) = 6x^2 - 12x$) and subtract.

$$\begin{array}{r|l} & x^2 + 6x \\ \hline x-2 & x^3 + 4x^2 - 5x - 14 \\ & -\underline{\left(x^3 - 2x^2\right)} \qquad \downarrow \\ & \qquad 6x^2 - 5x \quad \downarrow \\ & \qquad -\underline{\left(6x^2 - 12x\right)} \quad \downarrow \\ & \qquad\qquad 7x - 14 \end{array}$$

[4] In our opinion - you can judge for yourself.

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x - 14$) and subtract.

$$\begin{array}{r|l} & x^2 + 6x + 7 \\ \hline x-2 & x^3 + 4x^2 - 5x - 14 \\ & -\left(x^3 - 2x^2\right) \\ \cline{2-2} & \quad 6x^2 - 5x \\ & \quad -\left(6x^2 - 12x\right) \\ \cline{2-2} & \qquad 7x - 14 \\ & \qquad -(7x - 14) \\ \cline{2-2} & \qquad\qquad 0 \end{array}$$

In this case, we get a quotient of $x^2 + 6x + 7$ with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2 + 6x + 7\right) + 0 = x^3 + 6x^2 + 7x - 2x^2 - 12x - 14 = x^3 + 4x^2 - 5x - 14 \checkmark$$

2. To compute $(2t+7) \div (3t-4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t-4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$\begin{array}{r|l} & \qquad \frac{2}{3} \\ \hline 3t-4 & 2t + 7 \\ & -\left(2t - \frac{8}{3}\right) \\ \cline{2-2} & \qquad \frac{29}{3} \end{array}$$

Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t-4)\left(\frac{2}{3}\right) + \frac{29}{3} = 2t - \frac{8}{3} + \frac{29}{3} = 2t + \frac{21}{3} = 2t + 7 \checkmark$$

3. When we set-up the tableau for $\left(6y^2 - 1\right) \div (2y+5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2 - 1 = 6y^2 + 0y - 1$. We then proceed as before. Since $\frac{6y^2}{2y} = 3y$, $3y$ is the first term in our quotient. We multiply $(2y+5)$ times $3y$ and subtract

it from the dividend. We bring down the -1, and repeat.

$$\begin{array}{r|llll} & 3y & - & \frac{15}{2} & \\ \hline 2y+5 & 6y^2 + & 0y & - & 1 \\ & -(6y^2 + & 15y) & & \downarrow \\ \hline & & -15y & - & 1 \\ & & -\left(-15y - \frac{75}{2}\right) & & \\ \hline & & & & \frac{73}{2} \end{array}$$

Our answer is $3y - \frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y - \frac{15}{2}\right) + \frac{73}{2} = 6y^2 - 15y + 15y - \frac{75}{2} + \frac{73}{2} = 6y^2 - 1 \checkmark$$

4. For our last example, we need 'placeholders' for both the divisor $w^2 - \sqrt{2} = w^2 + 0w - \sqrt{2}$ and the dividend $w^3 = w^3 + 0w^2 + 0w + 0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2 + w\sqrt{2} + 0 = w\sqrt{2}$.

$$\begin{array}{r|llll} & & & w & \\ \hline w^2+0w-\sqrt{2} & w^3 & +0w^2 + & 0w & +0 \\ & -\left(w^3\right. & +0w^2 - & \left.w\sqrt{2}\right) & \downarrow \\ \hline & & 0w^2 + & w\sqrt{2} & +0 \end{array}$$

Since the degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), we are done.[5] Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2 - \sqrt{2}\right) w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

□

[5] Since $\frac{0w^2}{w^2} = 0$, we could proceed, write our quotient as $w + 0$, and move on... but even pedants have limits.

1.5.3 Exercises

In Exercises 1 - 15, perform the indicated operations and simplify.

1. $(4-3x)+(3x^2+2x+7)$
2. $t^2+4t-2(3-t)$
3. $q(200-3q)-(5q+500)$
4. $(3y-1)(2y+1)$
5. $\left(3-\frac{x}{2}\right)(2x+5)$
6. $-(4t+3)(t^2-2)$
7. $2w(w^3-5)(w^3+5)$
8. $(5a^2-3)(25a^4+15a^2+9)$
9. $(x^2-2x+3)(x^2+2x+3)$
10. $(\sqrt{7}-z)(\sqrt{7}+z)$
11. $(x-\sqrt[3]{5})^3$
12. $(x-\sqrt[3]{5})(x^2+x\sqrt[3]{5}+\sqrt[3]{25})$
13. $(w-3)^2-(w^2+9)$
14. $(x+h)^2-2(x+h)-(x^2-2x)$
15. $(x-[2+\sqrt{5}])(x-[2-\sqrt{5}])$

In Exercises 16 - 27, perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

16. $(5x^2-3x+1)\div(x+1)$
17. $(3y^2+6y-7)\div(y-3)$
18. $(6w-3)\div(2w+5)$
19. $(2x+1)\div(3x-4)$
20. $(t^2-4)\div(2t+1)$
21. $(w^3-8)\div(5w-10)$
22. $(2x^2-x+1)\div(3x^2+1)$
23. $(4y^4+3y^2+1)\div(2y^2-y+1)$
24. $w^4\div(w^3-2)$
25. $(5t^3-t+1)\div(t^2+4)$
26. $(t^3-4)\div(t-\sqrt[3]{4})$
27. $(x^2-2x-1)\div(x-[1-\sqrt{2}])$

In Exercises 28 - 33 verify the given formula by showing the left hand side of the equation simplifies to the right hand side of the equation.

28. **Perfect Cube:** $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
29. **Difference of Cubes:** $(a-b)(a^2+ab+b^2)=a^3-b^3$
30. **Sum of Cubes:** $(a+b)(a^2-ab+b^2)=a^3+b^3$
31. **Perfect Quartic:** $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$
32. **Difference of Quartics:** $(a-b)(a+b)(a^2+b^2)=a^4-b^4$
33. **Sum of Quartics:** $(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)=a^4+b^4$
34. With help from your classmates, determine under what conditions $(a+b)^2=a^2+b^2$. What about $(a+b)^3=a^3+b^3$? In general, when does $(a+b)^n=a^n+b^n$ for a natural number $n\geq 2$?

1.5.4 Answers

1. $3x^2 - x + 11$
2. $t^2 + 6t - 6$
3. $-3q^2 + 195q - 500$
4. $6y^2 + y - 1$
5. $-x^2 + \frac{7}{2}x + 15$
6. $-4t^3 - 3t^2 + 8t + 6$
7. $2w^7 - 50w$
8. $125a^6 - 27$
9. $x^4 + 2x^2 + 9$
10. $7 - z^2$
11. $x^3 - 3x^2\sqrt[3]{5} + 3x\sqrt[3]{25} - 5$
12. $x^3 - 5$
13. $-6w$
14. $h^2 + 2xh - 2h$
15. $x^2 - 4x - 1$
16. quotient: $5x - 8$, remainder: 9
17. quotient: $3y + 15$, remainder: 38
18. quotient: 3, remainder: 18
19. quotient: $\frac{2}{3}$, remainder: $\frac{11}{3}$
20. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$
21. quotient: $\frac{w^2}{5} + \frac{2w}{5} + \frac{4}{5}$, remainder: 0
22. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$
23. quotient: $2y^2 + y + 1$, remainder: 0
24. quotient: w, remainder: $2w$
25. quotient: $5t$, remainder: $-21t + 1$
26. quotient:[6] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0
27. quotient: $x - 1 - \sqrt{2}$, remainder: 0

[6] Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

1.6 Factoring

Now that we have reviewed the basics of polynomial arithmetic it's time to review the basic techniques of factoring polynomial expressions. Our goal is to apply these techniques to help us solve certain specialized classes of non-linear equations. Given that 'factoring' literally means to resolve a product into its factors, it is, in the purest sense, 'undoing' multiplication. If this sounds like division to you then you've been paying attention. Let's start with a numerical example.

Suppose we are asked to factor 16337. We could write $16337 = 16337 \cdot 1$, and while this is technically a factorization of 16337, it's probably not an answer the poser of the question would accept. Usually, when we're asked to factor a natural number, we are being asked to resolve it into to a product of so-called 'prime' numbers.[1] Recall that **prime numbers** are defined as natural numbers whose only (natural number) factors are themselves and 1. They are, in essence, the 'building blocks' of natural numbers as far as multiplication is concerned. Said differently, we can build - via multiplication - any natural number given enough primes. So how do we find the prime factors of 16337? We start by dividing each of the primes: 2, 3, 5, 7, etc., into 16337 until we get a remainder of 0. Eventually, we find that $16337 \div 17 = 961$ with a remainder of 0, which means $16337 = 17 \cdot 961$. So factoring and division are indeed closely related - factors of a number are precisely the divisors of that number which produce a zero remainder.[2] We continue our efforts to see if 961 can be factored down further, and we find that $961 = 31 \cdot 31$. Hence, 16337 can be 'completely factored' as $17 \cdot 31^2$. (This factorization is called the **prime factorization** of 16337.)

In factoring natural numbers, our building blocks are prime numbers, so to be completely factored means that every number used in the factorization of a given number is prime. One of the challenges when it comes to factoring polynomial expressions is to explain what it means to be 'completely factored'. In this section, our 'building blocks' for factoring polynomials are 'irreducible' polynomials as defined below.

Definition 1.14. A polynomial is said to be **irreducible** if it cannot be written as the product of polynomials of lower degree.

While Definition 1.14 seems straightforward enough, sometimes a greater level of specificity is required. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. While $x - \sqrt{3}$ and $x + \sqrt{3}$ are perfectly fine polynomials, factoring which requires irrational numbers is usually saved for a more advanced treatment of factoring.[3] For now, we will restrict ourselves to factoring using rational coefficients. So, while the polynomial $x^2 - 3$ can be factored using irrational numbers, it is called irreducible **over the rationals**, since there are no polynomials with *rational* coefficients of smaller degree which can be used to factor it.[4]

Since polynomials involve terms, the first step in any factoring strategy involves pulling out factors which are common to all of the terms. For example, in the polynomial $18x^2y^3 - 54x^3y^2 - 12xy^2$,

[1] As mentioned in Section 1.2, this is possible, in only one way, thanks to the Fundamental Theorem of Arithmetic.

[2] We'll refer back to this when we get to Section **??**.

[3] See Section **??**.

[4] If this isn't immediately obvious, don't worry - in some sense, it shouldn't be. We'll talk more about this later.

each coefficient is a multiple of 6 so we can begin the factorization as $6(3x^2y^3 - 9x^3y^2 - 2xy^2)$. The remaining coefficients: 3, 9 and 2, have no common factors so 6 was the greatest common factor. What about the variables? Each term contains an x, so we can factor an x from each term. When we do this, we are effectively dividing each term by x which means the exponent on x in each term is reduced by 1: $6x(3xy^3 - 9x^2y^2 - 2y^2)$. Next, we see that each term has a factor of y in it. In fact, each term has at least *two* factors of y in it, since the lowest exponent on y in each term is 2. This means that we can factor y^2 from each term. Again, factoring out y^2 from each term is tantamount to dividing each term by y^2 so the exponent on y in each term is reduced by *two*: $6xy^2(3xy - 9x^2 - 2)$. Just like we checked our division by multiplication in the previous section, we can check our factoring here by multiplication, too. $6xy^2(3xy - 9x^2 - 2) = (6xy^2)(3xy) - (6xy^2)(9x^2) - (6xy^2)(2) = 18x^2y^3 - 54x^3y^2 - 12xy^2$ ✓. We summarize how to find the Greatest Common Factor (G.C.F.) of a polynomial expression below.

Finding the G.C.F. of a Polynomial Expression

- If the coefficients are integers, find the G.C.F. of the coefficients.

  NOTE 1: If all of the coefficients are negative, consider the negative as part of the G.C.F..

  NOTE 2: If the coefficients involve fractions, get a common denominator, combine numerators, reduce to lowest terms and apply this step to the polynomial in the numerator.

- If a variable is common to all of the terms, the G.C.F. contains that variable to the smallest exponent which appears among the terms.

For example, to factor $-\frac{3}{5}z^3 - 6z^2$, we would first get a common denominator and factor as:

$$-\frac{3}{5}z^3 - 6z^2 = \frac{-3z^3 - 30z^2}{5} = \frac{-3z^2(z+10)}{5} = -\frac{3z^2(z+10)}{5}$$

We now list some common factoring formulas, each of which can be verified by multiplying out the right side of the equation. While they all should look familiar - this is a review section after all - some should look more familiar than others since they appeared as 'special product' formulas in the previous section.

Common Factoring Formulas

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

- **Difference of Two Squares:** $a^2 - b^2 = (a-b)(a+b)$

  NOTE: In general, the sum of squares, $a^2 + b^2$ is irreducible over the rationals.

- **Sum of Two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  NOTE: In general, $a^2 - ab + b^2$ is irreducible over the rationals.

- **Difference of Two Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  NOTE: In general, $a^2 + ab + b^2$ is irreducible over the rationals.

Our next example gives us practice with these formulas.

Example 1.6.1. Factor the following polynomials completely over the rationals. That is, write each polynomial as a product polynomials of lowest degree which are irreducible over the rationals.

1. $18x^2 - 48x + 32$
2. $64y^2 - 1$
3. $75t^4 + 30t^3 + 3t^2$
4. $w^4z - wz^4$
5. $81 - 16t^4$
6. $x^6 - 64$

Solution.

1. Our first step is to factor out the G.C.F. which in this case is 2. To match what is left with one of the special forms, we rewrite $9x^2 = (3x)^2$ and $16 = 4^2$. Since the 'middle' term is $-24x = -2(4)(3x)$, we see that we have a perfect square trinomial.

$$\begin{aligned} 18x^2 - 48x + 32 &= 2(9x^2 - 24x + 16) && \text{Factor out G.C.F.} \\ &= 2((3x)^2 - 2(4)(3x) + (4)^2) \\ &= 2(3x - 4)^2 && \text{Perfect Square Trinomial: } a = 3x,\ b = 4 \end{aligned}$$

   Our final answer is $2(3x - 4)^2$. To check, we multiply out $2(3x - 4)^2$ to show that it equals $18x^2 - 48x + 32$.

2. For $64y^2 - 1$, we note that the G.C.F. of the terms is just 1, so there is nothing (of substance) to factor out of both terms. Since $64y^2 - 1$ is the difference of two terms, one of which is a square, we look to the Difference of Squares Formula for inspiration. By identifying $64y^2 = (8y)^2$ and $1 = 1^2$, we get

$$\begin{aligned} 64y^2 - 1 &= (8y)^2 - 1^2 \\ &= (8y - 1)(8y + 1) && \text{Difference of Squares, } a = 8y,\ b = 1 \end{aligned}$$

   As before, we can check our answer by multiplying out $(8y - 1)(8y + 1)$ to show that it equals $64y^2 - 1$.

3. The G.C.F. of the terms in $75t^4 + 30t^3 + 3t^2$ is $3t^2$, so we factor that out first. We identify what remains as a perfect square trinomial:

$$\begin{aligned} 75t^4 + 30t^3 + 3t^2 &= 3t^2(25t^2 + 10t + 1) && \text{Factor out G.C.F.} \\ &= 3t^2((5t)^2 + 2(1)(5t) + 1^2) \\ &= 3t^2(5t + 1)^2 && \text{Perfect Square Trinomial, } a = 5t,\ b = 1 \end{aligned}$$

   Our final answer is $3t^2(5t + 1)^2$, which the reader is invited to check.

4. For $w^4z - wz^4$, we identify the G.C.F. as wz and once we factor it out a difference of cubes is revealed:

$$\begin{aligned} w^4z - wz^4 &= wz(w^3 - z^3) && \text{Factor out G.C.F.} \\ &= wz(w - z)(w^2 + wz + z^2) && \text{Difference of Cubes, } a = w,\ b = z \end{aligned}$$

   Our final answer is $wz(w - z)(w^2 + wz + z^2)$. The reader is strongly encouraged to multiply this out to see that it reduces to $w^4z - wz^4$.

5. The G.C.F. of the terms in $81 - 16t^4$ is just 1 so there is nothing of substance to factor out from both terms. With just a difference of two terms, we are limited to fitting this polynomial into either the Difference of Two Squares or Difference of Two Cubes formula. Since the variable here is t^4, and 4 is a multiple of 2, we can think of $t^4 = (t^2)^2$. This means that we can write $16t^4 = (4t^2)^2$ which is a perfect square. (Since 4 is not a multiple of 3, we cannot write t^4 as a perfect cube of a polynomial.) Identifying $81 = 9^2$ and $16t^4 = (4t^2)^2$, we apply the Difference of Squares Formula to get:

$$\begin{aligned} 81 - 16t^4 &= 9^2 - (4t^2)^2 \\ &= (9 - 4t^2)(9 + 4t^2) \quad \text{Difference of Squares, } a = 9,\ b = 4t^2 \end{aligned}$$

At this point, we have an opportunity to proceed further. Identifying $9 = 3^2$ and $4t^2 = (2t)^2$, we see that we have another difference of squares in the first quantity, which we can reduce. (The sum of two squares in the second quantity <u>cannot</u> be factored over the rationals.)

$$\begin{aligned} 81 - 16t^4 &= (9 - 4t^2)(9 + 4t^2) \\ &= (3^2 - (2t)^2)(9 + 4t^2) \\ &= (3 - 2t)(3 + 2t)(9 + 4t^2) \quad \text{Difference of Squares, } a = 3,\ b = 2t \end{aligned}$$

As always, the reader is encouraged to multiply out $(3-2t)(3+2t)(9+4t^2)$ to check the result.

6. With a G.C.F. of 1 and just two terms, $x^6 - 64$ is a candidate for both the Difference of Squares and the Difference of Cubes formulas. Notice that we can identify $x^6 = (x^3)^2$ and $64 = 8^2$ (both perfect squares), but also $x^6 = (x^2)^3$ and $64 = 4^3$ (both perfect cubes). If we follow the Difference of Squares approach, we get:

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - 8^2 \\ &= (x^3 - 8)(x^3 + 8) \quad \text{Difference of Squares, } a = x^3 \text{ and } b = 8 \end{aligned}$$

At this point, we have an opportunity to use both the Difference and Sum of Cubes formulas:

$$\begin{aligned} x^6 - 64 &= (x^3 - 2^3)(x^3 + 2^3) \\ &= (x - 2)(x^2 + 2x + 2^2)(x + 2)(x^2 - 2x + 2^2) \quad \text{Sum / Difference of Cubes, } a = x,\ b = 2 \\ &= (x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4) \quad \text{Rearrange factors} \end{aligned}$$

From this approach, our final answer is $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$.

Following the Difference of Cubes Formula approach, we get

$$\begin{aligned} x^6 - 64 &= (x^2)^3 - 4^3 \\ &= (x^2 - 4)((x^2)^2 + 4x^2 + 4^2) \quad \text{Difference of Cubes, } a = x^2,\ b = 4 \\ &= (x^2 - 4)(x^4 + 4x^2 + 16) \end{aligned}$$

At this point, we recognize $x^2 - 4$ as a difference of two squares:

$$\begin{aligned} x^6 - 64 &= (x^2 - 2^2)(x^4 + 4x^2 + 16) \\ &= (x - 2)(x + 2)(x^4 + 4x^2 + 16) \quad \text{Difference of Squares, } a = x,\ b = 2 \end{aligned}$$

Unfortunately, the remaining factor x^4+4x^2+16 is not a perfect square trinomial - the middle term would have to be $8x^2$ for this to work - so our final answer using this approach is $(x-2)(x+2)(x^4+4x^2+16)$. This isn't as factored as our result from the Difference of Squares approach which was $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. While it is true that $x^4+4x^2+16=(x^2-2x+4)(x^2+2x+4)$, there is no 'intuitive' way to motivate this factorization at this point.[5] The moral of the story? When given the option between using the Difference of Squares and Difference of Cubes, start with the Difference of Squares. Our final answer to this problem is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. The reader is strongly encouraged to show that this reduces down to x^6-64 after performing all of the multiplication. □

The formulas on page 71, while useful, can only take us so far, so we need to review some more advanced factoring strategies.

Advanced Factoring Formulas

- **'un-F.O.I.L.ing':** Given a trinomial Ax^2+Bx+C, try to reverse the F.O.I.L. process.

  That is, find a, b, c and d such that $Ax^2+Bx+C=(ax+b)(cx+d)$.

  NOTE: This means $ac=A$, $bd=C$ and $B=ad+bc$.

- **Factor by Grouping:** If the expression contains four terms with no common factors among the four terms, try 'factor by grouping':

$$ac+bc+ad+bd=(a+b)c+(a+b)d=(a+b)(c+d)$$

The techniques of 'un-F.O.I.L.ing' and 'factoring by grouping' are difficult to describe in general but should make sense to you with enough practice. Be forewarned - like all 'Rules of Thumb', these strategies work just often enough to be useful, but you can be sure there are exceptions which will defy any advice given here and will require some 'inspiration' to solve.[6] Even though Chapter **??** will give us more powerful factoring methods, we'll find that, in the end, there is no single algorithm for factoring which works for every polynomial. In other words, there will be times when you just have to try something and see what happens.

Example 1.6.2. Factor the following polynomials completely over the integers.[7]

1. x^2-x-6
2. $2t^2-11t+5$
3. $36-11y-12y^2$
4. $18xy^2-54xy-180x$
5. $2t^3-10t^2+3t-15$
6. x^4+4x^2+16

[5] Of course, this begs the question, "How do we know x^2-2x+4 and x^2+2x+4 are irreducible?" (We were told so on page 71, but no reason was given.) Stay tuned! We'll get back to this in due course.

[6] Jeff will be sure to pepper the Exercises with these.

[7] This means that all of the coefficients in the factors will be integers. In a rare departure from form, Carl decided to avoid fractions in this set of examples. Don't get complacent, though, because fractions will return with a vengeance soon enough.

Solution.

1. The G.C.F. of the terms $x^2 - x - 6$ is 1 and $x^2 - x - 6$ isn't a perfect square trinomial (Think about why not.) so we try to reverse the F.O.I.L. process and look for integers a, b, c and d such that $(ax+b)(cx+d) = x^2 - x - 6$. To get started, we note that $ac = 1$. Since a and c are meant to be integers, that leaves us with either a and c both being 1, or a and c both being -1. We'll go with $a = c = 1$, since we can factor[8] the negatives into our choices for b and d. This yields $(x+b)(x+d) = x^2 - x - 6$. Next, we use the fact that $bd = -6$. The product is negative so we know that one of b or d is positive and the other is negative. Since b and d are integers, one of b or d is ± 1 and the other is ∓ 6 OR one of b or d is ± 2 and the other is ∓ 3. After some guessing and checking,[9] we find that $x^2 - x - 6 = (x+2)(x-3)$.

2. As with the previous example, we check the G.C.F. of the terms in $2t^2 - 11t + 5$, determine it to be 1 and see that the polynomial doesn't fit the pattern for a perfect square trinomial. We now try to find integers a, b, c and d such that $(at+b)(ct+d) = 2t^2 - 11t + 5$. Since $ac = 2$, we have that one of a or c is 2, and the other is 1. (Once again, we ignore the negative options.) At this stage, there is nothing really distinguishing a from c so we choose $a = 2$ and $c = 1$. Now we look for b and d so that $(2t+b)(t+d) = 2t^2 - 11t + 5$. We know $bd = 5$ so one of b or d is ± 1 and the other ± 5. Given that bd is positive, b and d must have the same sign. The negative middle term $-11t$ guides us to guess $b = -1$ and $d = -5$ so that we get $(2t-1)(t-5) = 2t^2 - 11t + 5$. We verify our answer by multiplying.[10]

3. Once again, we check for a nontrivial G.C.F. and see if $36 - 11y - 12y^2$ fits the pattern of a perfect square. Twice disappointed, we rewrite $36 - 11y - 12y^2 = -12y^2 - 11y + 36$ for notational convenience. We now look for integers a, b, c and d such that $-12y^2 - 11y + 36 = (ay+b)(cy+d)$. Since $ac = -12$, we know that one of a or c is ± 1 and the other ± 12 OR one of them is ± 2 and the other is ± 6 OR one of them is ± 3 while the other is ± 4. Since their product is -12, however, we know one of them is positive, while the other is negative. To make matters worse, the constant term 36 has its fair share of factors, too. Our answers for b and d lie among the pairs ± 1 and ± 36, ± 2 and ± 18, ± 4 and ± 9, or ± 6. Since we know one of a or c will be negative, we can simplify our choices for b and d and just look at the positive possibilities. After some guessing and checking,[11] we find $(-3y+4)(4y+9) = -12y^2 - 11y + 36$.

4. Since the G.C.F. of the terms in $18xy^2 - 54xy - 180x$ is $18x$, we begin the problem by factoring it out first: $18xy^2 - 54xy - 180x = 18x(y^2 - 3y - 10)$. We now focus our attention on $y^2 - 3y - 10$. We can take a and c to both be 1 which yields $(y+b)(y+d) = y^2 - 3y - 10$. Our choices for b and d are among the factor pairs of -10: ± 1 and ± 10 or ± 2 and ± 5, where

[8] Pun intended!

[9] The authors have seen some strange gimmicks that allegedly help students with this step. We don't like them so we're sticking with good old-fashioned guessing and checking.

[10] That's the 'checking' part of 'guessing and checking'.

[11] Some of these guesses can be more 'educated' than others. Since the middle term is relatively 'small,' we don't expect the 'extreme' factors of 36 and 12 to appear, for instance.

one of b or d is positive and the other is negative. We find $(y-5)(y+2) = y^2 - 3y - 10$. Our final answer is $18xy^2 - 54xy - 180x = 18x(y-5)(y+2)$.

5. Since $2t^3 - 10t^2 - 3t + 15$ has four terms, we are pretty much resigned to factoring by grouping. The strategy here is to factor out the G.C.F. from two *pairs* of terms, and see if this reveals a common factor. If we group the first two terms, we can factor out a $2t^2$ to get $2t^3 - 10t^2 = 2t^2(t-5)$. We now try to factor something out of the last two terms that will leave us with a factor of $(t-5)$. Sure enough, we can factor out a -3 from both: $-3t+15 = -3(t-5)$. Hence, we get

$$2t^3 - 10t^2 - 3t + 15 = 2t^2(t-5) - 3(t-5) = (2t^2-3)(t-5)$$

Now the question becomes can we factor $2t^2 - 3$ over the integers? This would require integers a, b, c and d such that $(at+b)(ct+d) = 2t^2 - 3$. Since $ab = 2$ and $cd = -3$, we aren't left with many options - in fact, we really have only four choices: $(2t-1)(t+3)$, $(2t+1)(t-3)$, $(2t-3)(t+1)$ and $(2t+3)(t-1)$. None of these produces $2t^2 - 3$ - which means it's irreducible over the integers - thus our final answer is $(2t^2-3)(t-5)$.

6. Our last example, x^4+4x^2+16, is our old friend from Example 1.6.1. As noted there, it is not a perfect square trinomial, so we could try to reverse the F.O.I.L. process. This is complicated by the fact that our highest degree term is x^4, so we would have to look at factorizations of the form $(x+b)(x^3+d)$ as well as $(x^2+b)(x^2+d)$. We leave it to the reader to show that neither of those work. This is an example of where 'trying something' pays off. Even though we've stated that it is not a perfect square trinomial, it's pretty close. Identifying $x^4 = (x^2)^2$ and $16 = 4^2$, we'd have $(x^2+4)^2 = x^4 + 8x^2 + 16$, but instead of $8x^2$ as our middle term, we only have $4x^2$. We could add in the extra $4x^2$ we need, but to keep the balance, we'd have to subtract it off. Doing so produces and unexpected opportunity:

$$\begin{array}{rcll} x^4 + 4x^2 + 16 & = & x^4 + 4x^2 + 16 + (4x^2 - 4x^2) & \text{Adding and subtracting the same term} \\ & = & x^4 + 8x^2 + 16 - 4x^2 & \text{Rearranging terms} \\ & = & (x^2+4)^2 - (2x)^2 & \text{Factoring perfect square trinomial} \\ & = & [(x^2+4) - 2x][(x^2+4) + 2x] & \text{Difference of Squares: } a = (x^2+4),\ b = 2x \\ & = & (x^2 - 2x + 4)(x^2 + 2x + 4) & \text{Rearraging terms} \end{array}$$

We leave it to the reader to check that neither $x^2 - 2x + 4$ nor $x^2 + 2x + 4$ factor over the integers, so we are done. □

1.6.1 Solving Equations by Factoring

Many students wonder why they are forced to learn how to factor. Simply put, factoring is our main tool for solving the non-linear equations which arise in many of the applications of Mathematics.[12] We use factoring in conjunction with the Zero Product Property of Real Numbers which was first stated on page 19 and is given here again for reference.

[12] Also known as 'story problems' or 'real-world examples'.

The Zero Product Property of Real Numbers: If a and b are real numbers with $ab = 0$ then either $a = 0$ or $b = 0$ or both.

For example, consider the equation $6x^2 + 11x = 10$. To see how the Zero Product Property is used to help us solve this equation, we first set the equation equal to zero and then apply the techniques from Example 1.6.2:

$$\begin{array}{rcll} 6x^2 + 11x & = & 10 & \\ 6x^2 + 11x - 10 & = & 0 & \text{Subtract 10 from both sides} \\ (2x+5)(3x-2) & = & 0 & \text{Factor} \\ 2x + 5 = 0 & \text{or} & 3x - 2 = 0 & \text{Zero Product Property} \\ x = -\frac{5}{2} & \text{or} & x = \frac{2}{3} & a = 2x+5,\ b = 3x-2 \end{array}$$

The reader should check that both of these solutions satisfy the original equation.

It is critical that you see the importance of setting the expression equal to 0 before factoring. Otherwise, we'd get:

$$\begin{array}{rcll} 6x^2 + 11x & = & 10 & \\ x(6x+11) & = & 10 & \text{Factor} \end{array}$$

What we **cannot** deduce from this equation is that $x = 10$ or $6x + 11 = 10$ or that $x = 2$ and $6x + 11 = 5$, etc.. (It's wrong and you should feel bad if you do it.) It is precisely because 0 plays such a special role in the arithmetic of real numbers (as the Additive Identity) that we can assume a factor is 0 when the product is 0. No other real number has that ability.

We summarize the **correct** equation solving strategy below.

Strategy for Solving Non-linear Equations

1. Put all of the nonzero terms on one side of the equation so that the other side is 0.
2. Factor.
3. Use the Zero Product Property of Real Numbers and set each factor equal to 0.
4. Solve each of the resulting equations.

Let's finish the section with a collection of examples in which we use this strategy.

Example 1.6.3. Solve the following equations.

1. $3x^2 = 35 - 16x$
2. $t = \dfrac{1 + 4t^2}{4}$
3. $(y-1)^2 = 2(y-1)$
4. $\dfrac{w^4}{3} = \dfrac{8w^3 - 12}{12} - \dfrac{w^2 - 4}{4}$
5. $z(z(18z+9) - 50) = 25$
6. $x^4 - 8x^2 - 9 = 0$

Solution.

1. We begin by gathering all of the nonzero terms to one side getting 0 on the other and then we proceed to factor and apply the Zero Product Property.

$$\begin{array}{rclr} 3x^2 & = & 35 - 16x & \\ 3x^2 + 16x - 35 & = & 0 & \text{Add } 16x\text{, subtract } 35 \\ (3x-5)(x+7) & = & 0 & \text{Factor} \\ 3x - 5 = 0 & \text{or} & x + 7 = 0 & \text{Zero Product Property} \\ x = \frac{5}{3} & \text{or} & x = -7 & \end{array}$$

We check our answers by substituting each of them into the original equation. Plugging in $x = \frac{5}{3}$ yields $\frac{25}{3}$ on both sides while $x = -7$ gives 147 on both sides.

2. To solve $t = \frac{1+4t^2}{4}$, we first clear fractions then move all of the nonzero terms to one side of the equation, factor and apply the Zero Product Property.

$$\begin{array}{rclr} t & = & \dfrac{1+4t^2}{4} & \\ 4t & = & 1 + 4t^2 & \text{Clear fractions (multiply by 4)} \\ 0 & = & 1 + 4t^2 - 4t & \text{Subtract 4} \\ 0 & = & 4t^2 - 4t + 1 & \text{Rearrange terms} \\ 0 & = & (2t-1)^2 & \text{Factor (Perfect Square Trinomial)} \end{array}$$

At this point, we get $(2t-1)^2 = (2t-1)(2t-1) = 0$, so, the Zero Product Property gives us $2t - 1 = 0$ in both cases.[13] Our final answer is $t = \frac{1}{2}$, which we invite the reader to check.

3. Following the strategy outlined above, the first step to solving $(y-1)^2 = 2(y-1)$ is to gather the nonzero terms on one side of the equation with 0 on the other side and factor.

$$\begin{array}{rclr} (y-1)^2 & = & 2(y-1) & \\ (y-1)^2 - 2(y-1) & = & 0 & \text{Subtract } 2(y-1) \\ (y-1)[(y-1)-2] & = & 0 & \text{Factor out G.C.F.} \\ (y-1)(y-3) & = & 0 & \text{Simplify} \\ y - 1 = 0 & \text{or} & y - 3 = 0 & \\ y = 1 & \text{or} & y = 3 & \end{array}$$

Both of these answers are easily checked by substituting them into the original equation.

An alternative method to solving this equation is to begin by dividing both sides by $(y-1)$ to simplify things outright. As we saw in Example 1.3.1, however, whenever we divide by

[13]More generally, given a positive power p, the only solution to $X^p = 0$ is $X = 0$.

a variable quantity, we make the explicit assumption that this quantity is nonzero. Thus we must stipulate that $y-1 \neq 0$.

$$\begin{aligned} \frac{(y-1)^2}{(y-1)} &= \frac{2(y-1)}{(y-1)} && \text{Divide by } (y-1) \text{ - this assumes } (y-1) \neq 0 \\ y-1 &= 2 \\ y &= 3 \end{aligned}$$

Note that in this approach, we obtain the $y = 3$ solution, but we 'lose' the $y = 1$ solution. How did that happen? Assuming $y - 1 \neq 0$ is equivalent to assuming $y \neq 1$. This is an issue because $y = 1$ is a solution to the original equation and it was 'divided out' too early. The moral of the story? If you decide to divide by a variable expression, double check that you aren't excluding any solutions.[14]

4. Proceeding as before, we clear fractions, gather the nonzero terms on one side of the equation, have 0 on the other and factor.

$$\begin{aligned} \frac{w^4}{3} &= \frac{8w^3-12}{12} - \frac{w^2-4}{4} \\ 12\left(\frac{w^4}{3}\right) &= 12\left(\frac{8w^3-12}{12} - \frac{w^2-4}{4}\right) && \text{Multiply by 12} \\ 4w^4 &= (8w^3-12) - 3(w^2-4) && \text{Distribute} \\ 4w^4 &= 8w^3 - 12 - 3w^2 + 12 && \text{Distribute} \\ 0 &= 8w^3 - 12 - 3w^2 + 12 - 4w^4 && \text{Subtract } 4w^4 \\ 0 &= 8w^3 - 3w^2 - 4w^4 && \text{Gather like terms} \\ 0 &= w^2(8w - 3 - 4w^2) && \text{Factor out G.C.F.} \end{aligned}$$

At this point, we apply the Zero Product Property to deduce that $w^2 = 0$ or $8w - 3 - 4w^2 = 0$. From $w^2 = 0$, we get $w = 0$. To solve $8w - 3 - 4w^2 = 0$, we rearrange terms and factor: $-4w^2 + 8w - 3 = (2w-1)(-2w+3) = 0$. Applying the Zero Product Property again, we get $2w - 1 = 0$ (which gives $w = \frac{1}{2}$), or $-2w + 3 = 0$ (which gives $w = \frac{3}{2}$). Our final answers are $w = 0$, $w = \frac{1}{2}$ and $w = \frac{3}{2}$. The reader is encouraged to check each of these answers in the original equation. (You need the practice with fractions!)

5. For our next example, we begin by subtracting the 25 from both sides then work out the indicated operations before factoring by grouping.

$$\begin{aligned} z(z(18z+9)-50) &= 25 \\ z(z(18z+9)-50) - 25 &= 0 && \text{Subtract 25} \\ z(18z^2+9z-50) - 25 &= 0 && \text{Distribute} \\ 18z^3 + 9z^2 - 50z - 25 &= 0 && \text{Distribute} \\ 9z^2(2z+1) - 25(2z+1) &= 0 && \text{Factor} \\ (9z^2-25)(2z+1) &= 0 && \text{Factor} \end{aligned}$$

[14] You will see other examples throughout this text where dividing by a variable quantity does more harm than good. Keep this basic one in mind as you move on in your studies - it's a good cautionary tale.

At this point, we use the Zero Product Property and get $9z^2 - 25 = 0$ or $2z + 1 = 0$. The latter gives $z = -\frac{1}{2}$ whereas the former factors as $(3z - 5)(3z + 5) = 0$. Applying the Zero Product Property again gives $3z - 5 = 0$ (so $z = \frac{5}{3}$) or $3z + 5 = 0$ (so $z = -\frac{5}{3}$.) Our final answers are $z = -\frac{1}{2}$, $z = \frac{5}{3}$ and $z = -\frac{5}{3}$, each of which good fun to check.

6. The nonzero terms of the equation $x^4 - 8x^2 - 9 = 0$ are already on one side of the equation so we proceed to factor. This trinomial doesn't fit the pattern of a perfect square so we attempt to reverse the F.O.I.L.ing process. With an x^4 term, we have two possible forms to try: $(ax^2 + b)(cx^2 + d)$ and $(ax^3 + b)(cx + d)$. We leave it to you to show that $(ax^3 + b)(cx + d)$ does not work and we show that $(ax^2 + b)(cx^2 + d)$ does.

   Since the coefficient of x^4 is 1, we take $a = c = 1$. The constant term is -9 so we know b and d have opposite signs and our choices are limited to two options: either b and d come from ± 1 and ± 9 OR one is 3 while the other is -3. After some trial and error, we get $x^4 - 8x^2 - 9 = (x^2 - 9)(x^2 + 1)$. Hence $x^4 - 8x^2 - 9 = 0$ reduces to $(x^2 - 9)(x^2 + 1) = 0$. The Zero Product Property tells us that either $x^2 - 9 = 0$ or $x^2 + 1 = 0$. To solve the former, we factor: $(x - 3)(x + 3) = 0$, so $x - 3 = 0$ (hence, $x = 3$) or $x + 3 = 0$ (hence, $x = -3$). The equation $x^2 + 1 = 0$ has no (real) solution, since for any real number x, x^2 is always 0 or greater. Thus $x^2 + 1$ is always positive. Our final answers are $x = 3$ and $x = -3$. As always, the reader is invited to check both answers in the original equation. □

1.6.2 Exercises

In Exercises 1 - 30, factor completely over the integers. Check your answer by multiplication.

1. $2x - 10x^2$
2. $12t^5 - 8t^3$
3. $16xy^2 - 12x^2y$
4. $5(m+3)^2 - 4(m+3)^3$
5. $(2x-1)(x+3) - 4(2x-1)$
6. $t^2(t-5) + t - 5$
7. $w^2 - 121$
8. $49 - 4t^2$
9. $81t^4 - 16$
10. $9z^2 - 64y^4$
11. $(y+3)^2 - 4y^2$
12. $(x+h)^3 - (x+h)$
13. $y^2 - 24y + 144$
14. $25t^2 + 10t + 1$
15. $12x^3 - 36x^2 + 27x$
16. $m^4 + 10m^2 + 25$
17. $27 - 8x^3$
18. $t^6 + t^3$
19. $x^2 - 5x - 14$
20. $y^2 - 12y + 27$
21. $3t^2 + 16t + 5$
22. $6x^2 - 23x + 20$
23. $35 + 2m - m^2$
24. $7w - 2w^2 - 3$
25. $3m^3 + 9m^2 - 12m$
26. $x^4 + x^2 - 20$
27. $4(t^2-1)^2 + 3(t^2-1) - 10$
28. $x^3 - 5x^2 - 9x + 45$
29. $3t^2 + t - 3 - t^3$
30.[15] $y^4 + 5y^2 + 9$

In Exercises 31 - 45, find all rational number solutions. Check your answers.

31. $(7x+3)(x-5) = 0$
32. $(2t-1)^2(t+4) = 0$
33. $(y^2+4)(3y^2+y-10) = 0$
34. $4t = t^2$
35. $y + 3 = 2y^2$
36. $26x = 8x^2 + 21$
37. $16x^4 = 9x^2$
38. $w(6w+11) = 10$
39. $2w^2+5w+2 = -3(2w+1)$
40. $x^2(x-3) = 16(x-3)$
41. $(2t+1)^3 = (2t+1)$
42. $a^4 + 4 = 6 - a^2$
43. $\frac{8t^2}{3} = 2t + 3$
44. $\frac{x^3+x}{2} = \frac{x^2+1}{3}$
45. $\frac{y^4}{3} - y^2 = \frac{3}{2}(y^2+3)$
46. With help from your classmates, factor $4x^4 + 8x^2 + 9$.
47. With help from your classmates, find an equation which has 3, $-\frac{1}{2}$, and 117 as solutions.

[15] $y^4 + 5y^2 + 9 = (y^4 + 6y^2 + 9) - y^2$

1.6.3 Answers

1. $2x(1-5x)$
2. $4t^3(3t^2-2)$
3. $4xy(4y-3x)$
4. $-(m+3)^2(4m+7)$
5. $(2x-1)(x-1)$
6. $(t-5)(t^2+1)$
7. $(w-11)(w+11)$
8. $(7-2t)(7+2t)$
9. $(3t-2)(3t+2)(9t^2+4)$
10. $(3z-8y^2)(3z+8y^2)$
11. $-3(y-3)(y+1)$
12. $(x+h)(x+h-1)(x+h+1)$
13. $(y-12)^2$
14. $(5t+1)^2$
15. $3x(2x-3)^2$
16. $(m^2+5)^2$
17. $(3-2x)(9+6x+4x^2)$
18. $t^3(t+1)(t^2-t+1)$
19. $(x-7)(x+2)$
20. $(y-9)(y-3)$
21. $(3t+1)(t+5)$
22. $(2x-5)(3x-4)$
23. $(7-m)(5+m)$
24. $(-2w+1)(w-3)$
25. $3m(m-1)(m+4)$
26. $(x-2)(x+2)(x^2+5)$
27. $(2t-3)(2t+3)(t^2+1)$
28. $(x-3)(x+3)(x-5)$
29. $(t-3)(1-t)(1+t)$
30. $(y^2-y+3)(y^2+y+3)$
31. $x=-\frac{3}{7}$ or $x=5$
32. $t=\frac{1}{2}$ or $t=-4$
33. $y=\frac{5}{3}$ or $y=-2$
34. $t=0$ or $t=4$
35. $y=-1$ or $y=\frac{3}{2}$
36. $x=\frac{3}{2}$ or $x=\frac{7}{4}$
37. $x=0$ or $x=\pm\frac{3}{4}$
38. $w=-\frac{5}{2}$ or $w=\frac{2}{3}$
39. $w=-5$ or $w=-\frac{1}{2}$
40. $x=3$ or $x=\pm 4$
41. $t=-1$, $t=-\frac{1}{2}$, or $t=0$
42. $a=\pm 1$
43. $t=-\frac{3}{4}$ or $t=\frac{3}{2}$
44. $x=\frac{2}{3}$
45. $y=\pm 3$

1.7 Quadratic Equations

In Section 1.6.1, we reviewed how to solve basic non-linear equations by factoring. The astute reader should have noticed that all of the equations in that section were carefully constructed so that the polynomials could be factored using the integers. To demonstrate just how contrived the equations had to be, we can solve $2x^2+5x-3=0$ by factoring, $(2x-1)(x+3)=0$, from which we obtain $x=\frac{1}{2}$ and $x=-3$. If we change the 5 to a 6 and try to solve $2x^2+6x-3=0$, however, we find that this polynomial doesn't factor over the integers and we are stuck. It turns out that there are two real number solutions to this equation, but they are *irrational* numbers, and our aim in this section is to review the techniques which allow us to find these solutions.[1] In this section, we focus our attention on **quadratic** equations.

Definition 1.15. An equation is said to be **quadratic** in a variable X if it can be written in the form $AX^2+BX+C=0$ where A, B and C are expressions which do not involve X and $A \neq 0$.

Think of quadratic equations as equations that are one degree up from linear equations - instead of the highest power of X being just $X = X^1$, it's X^2. The simplest class of quadratic equations to solve are the ones in which $B=0$. In that case, we have the following.

Solving Quadratic Equations by Extracting Square Roots

If c is a real number with $c \geq 0$, the solutions to $X^2=c$ are $X=\pm\sqrt{c}$.

Note: If $c<0$, $X^2=c$ has no real number solutions.

There are a couple different ways to see why Extracting Square Roots works, both of which are demonstrated by solving the equation $x^2=3$. If we follow the procedure outlined in the previous section, we subtract 3 from both sides to get $x^2-3=0$ and we now try to factor x^2-3. As mentioned in the remarks following Definition 1.14, we could think of $x^2-3=x^2-(\sqrt{3})^2$ and apply the Difference of Squares formula to factor $x^2-3=(x-\sqrt{3})(x+\sqrt{3})$. We solve $(x-\sqrt{3})(x+\sqrt{3})=0$ by using the Zero Product Property as before by setting each factor equal to zero: $x-\sqrt{3}=0$ and $x+\sqrt{3}-0$. We get the answers $x=\pm\sqrt{3}$. In general, if $c \geq 0$, then $\sqrt{c}$ is a real number, so $x^2-c=x^2-(\sqrt{c})^2=(x-\sqrt{c})(x+\sqrt{c})$. Replacing the '3' with 'c' in the above discussion gives the general result.

Another way to view this result is to visualize 'taking the square root' of both sides: since $x^2=c$, $\sqrt{x^2}=\sqrt{c}$. How do we simplify $\sqrt{x^2}$? We have to exercise a bit of caution here. Note that $\sqrt{(5)^2}$ and $\sqrt{(-5)^2}$ both simplify to $\sqrt{25}=5$. In both cases, $\sqrt{x^2}$ returned a *positive* number, since the negative in -5 was 'squared away' *before* we took the square root. In other words, $\sqrt{x^2}$ is x if x is positive, or, if x is negative, we make x positive - that is, $\sqrt{x^2}=|x|$, the absolute value of x. So from $x^2=3$, we 'take the square root' of both sides of the equation to get $\sqrt{x^2}=\sqrt{3}$. This simplifies to $|x|=\sqrt{3}$, which by Theorem 1.3 is equivalent to $x=\sqrt{3}$ or $x=-\sqrt{3}$. Replacing the '3' in the previous argument with 'c,' gives the general result.

[1] While our discussion in this section departs from factoring, we'll see in Chapter **??** that the same correspondence between factoring and solving equations holds whether or not the polynomial factors over the integers.

As you might expect, Extracting Square Roots can be applied to more complicated equations. Consider the equation below. We can solve it by Extracting Square Roots provided we first isolate the perfect square quantity:

$$\begin{aligned}
2\left(x+\frac{3}{2}\right)^2-\frac{15}{2} &= 0 & \\
2\left(x+\frac{3}{2}\right)^2 &= \frac{15}{2} & \text{Add } \frac{15}{2} \\
\left(x+\frac{3}{2}\right)^2 &= \frac{15}{4} & \text{Divide by 2} \\
x+\frac{3}{2} &= \pm\sqrt{\frac{15}{4}} & \text{Extract Square Roots} \\
x+\frac{3}{2} &= \pm\frac{\sqrt{15}}{2} & \text{Property of Radicals} \\
x &= -\frac{3}{2}\pm\frac{\sqrt{15}}{2} & \text{Subtract } \frac{3}{2} \\
x &= -\frac{3\pm\sqrt{15}}{2} & \text{Add fractions}
\end{aligned}$$

Let's return to the equation $2x^2+6x-3=0$ from the beginning of the section. We leave it to the reader to show that

$$2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}=2x^2+6x-3.$$

(Hint: Expand the left side.) In other words, we can solve $2x^2+6x-3=0$ by *transforming* into an equivalent equation. This process, you may recall, is called 'Completing the Square.' We'll revisit Completing the Square in Section **??** in more generality and for a different purpose but for now we revisit the steps needed to complete the square to solve a quadratic equation.

Solving Quadratic Equations: Completing the Square

To solve a quadratic equation $AX^2+BX+C=0$ by Completing the Square:

1. Subtract the constant C from both sides.
2. Divide both sides by A, the coefficient of X^2. (Remember: $A\neq 0$.)
3. Add $\left(\frac{B}{2A}\right)^2$ to both sides of the equation. (That's half the coefficient of X, squared.)
4. Factor the left hand side of the equation as $\left(X+\frac{B}{2A}\right)^2$.
5. Extract Square Roots.
6. Subtract $\frac{B}{2A}$ from both sides.

To refresh our memories, we apply this method to solve $3x^2 - 24x + 5 = 0$:

$$\begin{aligned}
3x^2 - 24x + 5 &= 0 && \\
3x^2 - 24x &= -5 && \text{Subtract } C = 5 \\
x^2 - 8x &= -\frac{5}{3} && \text{Divide by } A = 3 \\
x^2 - 8x + 16 &= -\frac{5}{3} + 16 && \text{Add } \left(\tfrac{B}{2A}\right)^2 = (-4)^2 = 16 \\
(x-4)^2 &= \frac{43}{3} && \text{Factor: Perfect Square Trinomial} \\
x - 4 &= \pm\sqrt{\frac{43}{3}} && \text{Extract Square Roots} \\
x &= 4 \pm \sqrt{\frac{43}{3}} && \text{Add } 4
\end{aligned}$$

At this point, we use properties of fractions and radicals to 'rationalize' the denominator:[2]

$$\sqrt{\frac{43}{3}} = \sqrt{\frac{43 \cdot 3}{3 \cdot 3}} = \frac{\sqrt{129}}{\sqrt{9}} = \frac{\sqrt{129}}{3}$$

We can now get a common (integer) denominator which yields:

$$x = 4 \pm \sqrt{\frac{43}{3}} = 4 \pm \frac{\sqrt{129}}{3} = \frac{12 \pm \sqrt{129}}{3}$$

The key to Completing the Square is that the procedure always produces a perfect square trinomial. To see why this works *every single time*, we start with $AX^2 + BX + C = 0$ and follow the procedure:

$$\begin{aligned}
AX^2 + BX + C &= 0 && \\
AX^2 + BX &= -C && \text{Subtract } C \\
X^2 + \frac{BX}{A} &= -\frac{C}{A} && \text{Divide by } A \neq 0 \\
X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 &= -\frac{C}{A} + \left(\frac{B}{2A}\right)^2 && \text{Add } \left(\frac{B}{2A}\right)^2
\end{aligned}$$

(Hold onto the line above for a moment.) Here's the heart of the method - we need to show that

$$X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 = \left(X + \frac{B}{2A}\right)^2$$

To show this, we start with the right side of the equation and apply the Perfect Square Formula from Theorem 1.7

$$\left(X + \frac{B}{2A}\right)^2 = X^2 + 2\left(\frac{B}{2A}\right)X + \left(\frac{B}{2A}\right)^2 = X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 \ \checkmark$$

[2]Recall that this means we want to get a denominator with rational (more specifically, integer) numbers.

With just a few more steps we can solve the general equation $AX^2 + BX + C = 0$ so let's pick up the story where we left off. (The line on the previous page we told you to hold on to.)

$$\begin{array}{rcll}
X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 & = & -\frac{C}{A} + \left(\frac{B}{2A}\right)^2 & \\
\left(X + \frac{B}{2A}\right)^2 & = & -\frac{C}{A} + \frac{B^2}{4A^2} & \text{Factor: Perfect Square Trinomial} \\
\left(X + \frac{B}{2A}\right)^2 & = & -\frac{4AC}{4A^2} + \frac{B^2}{4A^2} & \text{Get a common denominator} \\
\left(X + \frac{B}{2A}\right)^2 & = & \frac{B^2 - 4AC}{4A^2} & \text{Add fractions} \\
X + \frac{B}{2A} & = & \pm\sqrt{\frac{B^2 - 4AC}{4A^2}} & \text{Extract Square Roots} \\
X + \frac{B}{2A} & = & \pm\frac{\sqrt{B^2 - 4AC}}{2A} & \text{Properties of Radicals} \\
X & = & -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} & \text{Subtract } \frac{B}{2A} \\
X & = & \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} & \text{Add fractions.}
\end{array}$$

Lo and behold, we have derived the legendary **Quadratic Formula**!

Theorem 1.9. Quadratic Formula: The solution to $AX^2 + BX + C = 0$ with $A \neq 0$ is:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We can check our earlier solutions to $2x^2 + 6x - 3 = 0$ and $3x^2 - 24x + 5 = 0$ using the Quadratic Formula. For $2x^2 + 6x - 3 = 0$, we identify $A = 2$, $B = 6$ and $C = -3$. The quadratic formula gives:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-3)}}{2(2)} - \frac{-6 \pm \sqrt{36 + 24}}{4} = \frac{-6 \pm \sqrt{60}}{4}$$

Using properties of radicals ($\sqrt{60} = 2\sqrt{15}$), this reduces to $\frac{2(-3 \pm \sqrt{15})}{4} = \frac{-3 \pm \sqrt{15}}{2}$. We leave it to the reader to show these two answers are the same as $-\frac{3 \pm \sqrt{15}}{2}$, as required.[3]
For $3x^2 - 24x + 5 = 0$, we identify $A = 3$, $B = -24$ and $C = 5$. Here, we get:

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(3)(5)}}{2(3)} = \frac{24 \pm \sqrt{516}}{6}$$

Since $\sqrt{516} = 2\sqrt{129}$, this reduces to $x = \frac{12 \pm \sqrt{129}}{3}$.

[3] Think about what $-(3 \pm \sqrt{15})$ is really telling you.

It is worth noting that the Quadratic Formula applies to all quadratic equations - even ones we could solve using other techniques. For example, to solve $2x^2+5x-3=0$ we identify $A=2$, $B=5$ and $C=-3$. This yields:

$$x = \frac{-5 \pm \sqrt{5^2-4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

At this point, we have $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = \frac{-12}{4} = -3$ - the same two answers we obtained factoring. We can also use it to solve $x^2 = 3$, if we wanted to. From $x^2 - 3 = 0$, we have $A = 1$, $B = 0$ and $C = -3$. The Quadratic Formula produces

$$x = \frac{-0 \pm \sqrt{0^2-4(1)(3)}}{2(1)} = \frac{\pm\sqrt{12}}{2} = \pm\frac{2\sqrt{3}}{2} = \pm\sqrt{3}$$

As this last example illustrates, while the Quadratic Formula *can* be used to solve every quadratic equation, that doesn't mean it *should* be used. Many times other methods are more efficient. We now provide a more comprehensive approach to solving Quadratic Equations.

Strategies for Solving Quadratic Equations

- If the variable appears in the squared term only, isolate it and Extract Square Roots.
- Otherwise, put the nonzero terms on one side of the equation so that the other side is 0.
  - Try factoring.
  - If the expression doesn't factor easily, use the Quadratic Formula.

The reader is encouraged to pause for a moment to think about why 'Completing the Square' doesn't appear in our list of strategies despite the fact that we've spent the majority of the section so far talking about it.[4] Let's get some practice solving quadratic equations, shall we?

Example 1.7.1. Find all real number solutions to the following equations.

1. $3-(2w-1)^2=0$
2. $5x-x(x-3)=7$
3. $(y-1)^2 = 2 - \frac{y+2}{3}$
4. $5(25-21x) = \frac{59}{4} - 25x^2$
5. $-4.9t^2 + 10t\sqrt{3} + 2 = 0$
6. $2x^2 = 3x^4 - 6$

Solution.

1. Since $3-(2w-1)^2=0$ contains a perfect square, we isolate it first then extract square roots:

$$\begin{array}{rcll} 3-(2w-1)^2 & = & 0 & \\ 3 & = & (2w-1)^2 & \text{Add } (2w-1)^2 \\ \pm\sqrt{3} & = & 2w-1 & \text{Extract Square Roots} \\ 1\pm\sqrt{3} & = & 2w & \text{Add 1} \\ \frac{1\pm\sqrt{3}}{2} & = & w & \text{Divide by 2} \end{array}$$

[4]Unacceptable answers include "Jeff and Carl are mean" and "It was one of Carl's Pedantic Rants".

We find our two answers $w = \frac{1 \pm \sqrt{3}}{2}$. The reader is encouraged to check both answers by substituting each into the original equation.[5]

2. To solve $5x - x(x-3) = 7$, we begin performing the indicated operations and getting one side equal to 0.

$$\begin{array}{rclr} 5x - x(x-3) & = & 7 & \\ 5x - x^2 + 3x & = & 7 & \text{Distribute} \\ -x^2 + 8x & = & 7 & \text{Gather like terms} \\ -x^2 + 8x - 7 & = & 0 & \text{Subtract 7} \end{array}$$

At this point, we attempt to factor and find $-x^2 + 8x - 7 = (x-1)(-x+7)$. Using the Zero Product Property, we get $x - 1 = 0$ or $-x + 7 = 0$. Our answers are $x = 1$ or $x = 7$, both of which are easy to check.

3. Even though we have a perfect square in $(y-1)^2 = 2 - \frac{y+2}{3}$, Extracting Square Roots won't help matters since we have a y on the other side of the equation. Our strategy here is to perform the indicated operations (and clear the fraction for good measure) and get 0 on one side of the equation.

$$\begin{array}{rclr} (y-1)^2 & = & 2 - \dfrac{y+2}{3} & \\ y^2 - 2y + 1 & = & 2 - \dfrac{y+2}{3} & \text{Perfect Square Trinomial} \\ 3(y^2 - 2y + 1) & = & 3\left(2 - \dfrac{y+2}{3}\right) & \text{Multiply by 3} \\ 3y^2 - 6y + 3 & = & 6 - 3\left(\dfrac{y+2}{3}\right) & \text{Distribute} \\ 3y^2 - 6y + 3 & = & 6 - (y+2) & \\ 3y^2 - 6y + 3 - 6 + (y+2) & = & 0 & \text{Subtract 6, Add } (y+2) \\ 3y^2 - 5y - 1 & = & 0 & \end{array}$$

A cursory attempt at factoring bears no fruit, so we run this through the Quadratic Formula with $A = 3$, $B = -5$ and $C = -1$.

$$\begin{array}{rcl} y & = & \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} \\ y & = & \dfrac{5 \pm \sqrt{25 + 12}}{6} \\ y & = & \dfrac{5 \pm \sqrt{37}}{6} \end{array}$$

Since 37 is prime, we have no way to reduce $\sqrt{37}$. Thus, our final answers are $y = \frac{5 \pm \sqrt{37}}{6}$. The reader is encouraged to supply the details of the challenging verification of the answers.

[5] It's excellent practice working with radicals fractions so we really, *really* want you to take the time to do it.

4. We proceed as before; our aim is to gather the nonzero terms on one side of the equation.

$$\begin{array}{rcll}
5(25-21x) & = & \dfrac{59}{4}-25x^2 & \\
125-105x & = & \dfrac{59}{4}-25x^2 & \text{Distribute} \\
4(125-105x) & = & 4\left(\dfrac{59}{4}-25x^2\right) & \text{Multiply by 4} \\
500-420x & = & 59-100x^2 & \text{Distribute} \\
500-420x-59+100x^2 & = & 0 & \text{Subtract 59, Add } 100x^2 \\
100x^2-420x+441 & = & 0 & \text{Gather like terms}
\end{array}$$

With highly composite numbers like 100 and 441, factoring seems inefficient at best,[6] so we apply the Quadratic Formula with $A = 100$, $B = -420$ and $C = 441$:

$$\begin{array}{rcl}
x & = & \dfrac{-(-420)\pm\sqrt{(-420)^2-4(100)(441)}}{2(100)} \\
 & = & \dfrac{420\pm\sqrt{176000-176400}}{200} \\
 & = & \dfrac{420\pm\sqrt{0}}{200} \\
 & = & \dfrac{420\pm 0}{200} \\
 & = & \dfrac{420}{200} \\
 & = & \dfrac{21}{10}
\end{array}$$

To our surprise and delight we obtain just one answer, $x = \frac{21}{10}$.

5. Our next equation $-4.9t^2 + 10t\sqrt{3} + 2 = 0$, already has 0 on one side of the equation, but with coefficients like -4.9 and $10\sqrt{3}$, factoring with integers is not an option. We could make things a *bit* easier on the eyes by clearing the decimal (by multiplying through by 10) to get $-49t^2+100t\sqrt{3}+20 = 0$ but we simply cannot rid ourselves of the irrational number $\sqrt{3}$. The Quadratic Formula is our only recourse. With $A = -49$, $B = 100\sqrt{3}$ and $C = 20$ we get:

[6] This is actually the Perfect Square Trinomial $(10x - 21)^2$.

$$\begin{aligned}
t &= \frac{-100\sqrt{3} \pm \sqrt{(100\sqrt{3})^2 - 4(-49)(20)}}{2(-49)} \\
&= \frac{-100\sqrt{3} \pm \sqrt{30000 + 3920}}{-98} \\
&= \frac{-100\sqrt{3} \pm \sqrt{33920}}{-98} \\
&= \frac{-100\sqrt{3} \pm 8\sqrt{530}}{-98} \\
&= \frac{2(-50\sqrt{3} \pm 4\sqrt{530})}{2(-49)} \\
&= \frac{-50\sqrt{3} \pm 4\sqrt{530}}{-49} && \text{Reduce} \\
&= \frac{-(-50\sqrt{3} \pm 4\sqrt{530})}{49} && \text{Properties of Negatives} \\
&= \frac{50\sqrt{3} \mp 4\sqrt{530}}{49} && \text{Distribute}
\end{aligned}$$

You'll note that when we 'distributed' the negative in the last step, we changed the '$\pm$' to a '$\mp$.' While this is technically correct, at the end of the day both symbols mean 'plus or minus',[7] so we can write our answers as $t = \frac{50\sqrt{3} \pm 4\sqrt{530}}{49}$. Checking these answers are a true test of arithmetic mettle.

6. At first glance, the equation $2x^2 = 3x^4 - 6$ seems misplaced. The highest power of the variable x here is 4, not 2, so this equation isn't a quadratic equation - at least not in terms of the variable x. It is, however, an example of an equation that is quadratic 'in disguise.'[8] We introduce a new variable u to help us see the pattern - specifically we let $u = x^2$. Thus $u^2 = (x^2)^2 = x^4$. So in terms of the variable u, the equation $2x^2 = 3x^4 - 6$ is $2u = 3u^2 - 6$. The latter is a quadratic equation, which we can solve using the usual techniques:

$$\begin{aligned}
2u &= 3u^2 - 6 \\
0 &= 3u^2 - 2u - 6 && \text{Subtract } 2u
\end{aligned}$$

After a few attempts at factoring, we resort to the Quadratic Formula with $A = 3$, $B = -2$,

[7] There are instances where we need both symbols, however. For example, the Sum and Difference of Cubes Formulas (page 71) can be written as a single formula: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. In this case, all of the 'top' symbols are read to give the sum formula; the 'bottom' symbols give the difference formula.

[8] More formally, **quadratic in form.** Carl likes 'Quadratics in Disguise' since it reminds him of the tagline of one of his beloved childhood cartoons and toy lines.

$C = -6$ and get:

$$\begin{aligned}
u &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} \\
&= \frac{2 \pm \sqrt{4+72}}{6} \\
&= \frac{2 \pm \sqrt{76}}{6} \\
&= \frac{2 \pm \sqrt{4 \cdot 19}}{6} \\
&= \frac{2 \pm 2\sqrt{19}}{6} && \text{Properties of Radicals} \\
&= \frac{2(1 \pm \sqrt{19})}{2(3)} && \text{Factor} \\
&= \frac{1 \pm \sqrt{19}}{3} && \text{Reduce}
\end{aligned}$$

We've solved the equation for u, but what we still need to solve the original equation[9] - which means we need to find the corresponding values of x. Since $u = x^2$, we have two equations:

$$x^2 = \frac{1+\sqrt{19}}{3} \quad \text{or} \quad x^2 = \frac{1-\sqrt{19}}{3}$$

We can solve the first equation by extracting square roots to get $x = \pm\sqrt{\frac{1+\sqrt{19}}{3}}$. The second equation, however, has no real number solutions because $\frac{1-\sqrt{19}}{3}$ is a negative number. For our final answers we can rationalize the denominator[10] to get:

$$x = \pm\sqrt{\frac{1+\sqrt{19}}{3}} = \pm\sqrt{\frac{1+\sqrt{19}}{3} \cdot \frac{3}{3}} = \pm\frac{\sqrt{3+3\sqrt{19}}}{3}$$

As with the previous exercise, the very challenging check is left to the reader. □

Our last example above, the 'Quadratic in Disguise', hints that the Quadratic Formula is applicable to a wider class of equations than those which are strictly quadratic. We give some general guidelines to recognizing these beasts in the wild on the next page.

[9]Or, you've solved the equation for 'you' (u), now you have to solve it for your instructor (x).

[10]We'll say more about this technique in Section 1.9.

Identifying Quadratics in Disguise

An equation is a 'Quadratic in Disguise' if it can be written in the form: $AX^{2m} + BX^m + C = 0$. In other words:

- There are exactly three terms, two with variables and one constant term.
- The exponent on the variable in one term is *exactly twice* the variable on the other term.

To transform a Quadratic in Disguise to a quadratic equation, let $u = X^m$ so $u^2 = (X^m)^2 = X^{2m}$. This transforms the equation into $Au^2 + Bu + C = 0$.

For example, $3x^6 - 2x^3 + 1 = 0$ is a Quadratic in Disguise, since $6 = 2 \cdot 3$. If we let $u = x^3$, we get $u^2 = (x^3)^2 = x^6$, so the equation becomes $3u^2 - 2u + 1 = 0$. However, $3x^6 - 2x^2 + 1 = 0$ is *not* a Quadratic in Disguise, since $6 \neq 2 \cdot 2$. The substitution $u = x^2$ yields $u^2 = (x^2)^2 = x^4$, not x^6 as required. We'll see more instances of 'Quadratics in Disguise' in later sections.

We close this section with a review of the **discriminant** of a quadratic equation as defined below.

Definition 1.16. The Discriminant: Given a quadratic equation $AX^2 + BX + C = 0$, the quantity $B^2 - 4AC$ is called the **discriminant** of the equation.

The discriminant is the radicand of the square root in the quadratic formula:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

It *discriminates* between the nature and number of solutions we get from a quadratic equation. The results are summarized below.

Theorem 1.10. Discriminant Theorem: Given a Quadratic Equation $AX^2 + BX + C = 0$, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one repeated real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B \pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are no real solutions.

For example, $x^2 + x - 1 = 0$ has two real number solutions since the discriminant works out to be $(1)^2 - 4(1)(-1) = 5 > 0$. This results in a $\pm\sqrt{5}$ in the Quadratic Formula, generating two different answers. On the other hand, $x^2 + x + 1 = 0$ has no real solutions since here, the discriminant is $(1)^2 - 4(1)(1) = -3 < 0$ which generates a $\pm\sqrt{-3}$ in the Quadratic Formula. The equation $x^2 + 2x + 1 = 0$ has discriminant $(2)^2 - 4(1)(1) = 0$ so in the Quadratic Formula we get a $\pm\sqrt{0} = 0$ thereby generating just one solution. More can be said as well. For example, the discriminant of $6x^2 - x - 40 = 0$ is 961. This is a perfect square, $\sqrt{961} = 31$, which means our solutions are rational numbers. When our solutions are rational numbers, the quadratic actually factors nicely. In our example $6x^2 - x - 40 = (2x + 5)(3x - 8)$. Admittedly, if you've already computed the discriminant, you're most of the way done with the problem and probably wouldn't take the time to experiment with factoring the quadratic at this point – but we'll see another use for this analysis of the discriminant in the next section.[11]

[11] Specifically in Example 1.8.1.

1.7.1 Exercises

In Exercises 1 - 21, find all real solutions. Check your answers, as directed by your instructor.

1. $3\left(x-\frac{1}{2}\right)^2=\frac{5}{12}$
2. $4-(5t+3)^2=3$
3. $3(y^2-3)^2-2=10$
4. $x^2+x-1=0$
5. $3w^2=2-w$
6. $y(y+4)=1$
7. $\frac{z}{2}=4z^2-1$
8. $0.1v^2+0.2v=0.3$
9. $x^2=x-1$
10. $3-t=2(t+1)^2$
11. $(x-3)^2=x^2+9$
12. $(3y-1)(2y+1)=5y$
13. $w^4+3w^2-1=0$
14. $2x^4+x^2=3$
15. $(2-y)^4=3(2-y)^2+1$
16. $3x^4+6x^2=15x^3$
17. $6p+2=p^2+3p^3$
18. $10v=7v^3-v^5$
19. $y^2-\sqrt{8}y=\sqrt{18}y-1$
20. $x^2\sqrt{3}=x\sqrt{6}+\sqrt{12}$
21. $\frac{v^2}{3}=\frac{v\sqrt{3}}{2}+1$

In Exercises 22 - 27, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

22. $5.54^2+b^2=36$
23. $\pi r^2=37$
24. $54=8r\sqrt{2}+\pi r^2$
25. $-4.9t^2+100t=410$
26. $x^2=1.65(3-x)^2$
27. $(0.5+2A)^2=0.7(0.1-A)^2$

In Exercises 28 - 30, use Theorem 1.3 along with the techniques in this section to find all real solutions to the following.

28. $|x^2-3x|=2$
29. $|2x-x^2|=|2x-1|$
30. $|x^2-x+3|=|4-x^2|$
31. Prove that for every nonzero number p, $x^2+xp+p^2=0$ has no real solutions.
32. Solve for t: $-\frac{1}{2}gt^2+vt+h=0$. Assume $g>0$, $v\geq 0$ and $h\geq 0$.

1.7.2 Answers

1. $x = \dfrac{3 \pm \sqrt{5}}{6}$
2. $t = -\dfrac{4}{5}, -\dfrac{2}{5}$
3. $y = \pm 1, \pm\sqrt{5}$
4. $x = \dfrac{-1 \pm \sqrt{5}}{2}$
5. $w = -1, \dfrac{2}{3}$
6. $y = -2 \pm \sqrt{5}$
7. $z = \dfrac{1 \pm \sqrt{65}}{16}$
8. $v = -3, 1$
9. No real solution.
10. $t = \dfrac{-5 \pm \sqrt{33}}{4}$
11. $x = 0$
12. $y = \dfrac{2 \pm \sqrt{10}}{6}$
13. $w = \pm\sqrt{\dfrac{\sqrt{13} - 3}{2}}$
14. $x = \pm 1$
15. $y = \dfrac{4 \pm \sqrt{6 + 2\sqrt{13}}}{2}$
16. $x = 0, \dfrac{5 \pm \sqrt{17}}{2}$
17. $p = -\dfrac{1}{3}, \pm\sqrt{2}$
18. $v = 0, \pm\sqrt{2}, \pm\sqrt{5}$
19. $y = \dfrac{5\sqrt{2} \pm \sqrt{46}}{2}$
20. $x = \dfrac{\sqrt{2} \pm \sqrt{10}}{2}$
21. $v = -\dfrac{\sqrt{3}}{2}, 2\sqrt{3}$
22. $b = \pm\dfrac{\sqrt{13271}}{50} \approx \pm 2.30$
23. $r = \pm\sqrt{\dfrac{37}{\pi}} \approx \pm 3.43$
24. $r = \dfrac{-4\sqrt{2} \pm \sqrt{54\pi + 32}}{\pi}$, $r \approx -6.32, 2.72$
25. $t = \dfrac{500 \pm 10\sqrt{491}}{49}$, $t \approx 5.68, 14.73$
26. $x = \dfrac{99 \pm 6\sqrt{165}}{13}$, $x \approx 1.69, 13.54$
27. $A = \dfrac{-107 \pm 7\sqrt{70}}{330}$, $A \approx -0.50, -0.15$
28. $x = 1, 2, \dfrac{3 \pm \sqrt{17}}{2}$
29. $x = \pm 1, 2 \pm \sqrt{3}$
30. $x = -\dfrac{1}{2}, 1, 7$
31. The discriminant is: $D = p^2 - 4p^2 = -3p^2 < 0$. Since $D < 0$, there are no real solutions.
32. $t = \dfrac{v \pm \sqrt{v^2 + 2gh}}{g}$

1.8 Rational Expressions and Equations

We now turn our attention to rational expressions - that is, algebraic fractions - and equations which contain them. The reader is encouraged to keep in mind the properties of fractions listed on page 20 because we will need them along the way. Before we launch into reviewing the basic arithmetic operations of rational expressions, we take a moment to review how to simplify them properly. As with numeric fractions, we 'cancel common *factors*,' not common *terms*. That is, in order to simplify rational expressions, we first *factor* the numerator and denominator. For example:

$$\frac{x^4 + 5x^3}{x^3 - 25x} \neq \frac{x^4 + 5\cancel{x^3}}{\cancel{x^3} - 25x}$$

but, rather

$$\begin{aligned}
\frac{x^4 + 5x^3}{x^3 - 25x} &= \frac{x^3(x+5)}{x(x^2-25)} && \text{Factor G.C.F.} \\
&= \frac{x^3(x+5)}{x(x-5)(x+5)} && \text{Difference of Squares} \\
&= \frac{\cancelto{x^2}{x^3}\cancel{(x+5)}}{\cancel{x}(x-5)\cancel{(x+5)}} && \text{Cancel common factors} \\
&= \frac{x^2}{x-5}
\end{aligned}$$

This equivalence holds provided the factors being canceled aren't 0. Since a factor of x and a factor of $x + 5$ were canceled, $x \neq 0$ and $x + 5 \neq 0$, so $x \neq -5$. We usually stipulate this as:

$$\frac{x^4 + 5x^3}{x^3 - 25x} = \frac{x^2}{x-5}, \qquad \text{provided } x \neq 0,\ x \neq -5$$

While we're talking about common mistakes, please notice that

$$\frac{5}{x^2+9} \neq \frac{5}{x^2} + \frac{5}{9}$$

Just like their numeric counterparts, you don't add algebraic fractions by *adding denominators* of fractions with *common numerators* - it's the other way around:[1]

$$\frac{x^2+9}{5} = \frac{x^2}{5} + \frac{9}{5}$$

It's time to review the basic arithmetic operations with rational expressions.

[1] One of the most common errors students make on college Mathematics placement tests is that they forget how to add algebraic fractions correctly. This places many students into remedial classes even though they are probably ready for college-level Math. We urge you to really study this section with great care so that you don't fall into that trap.

Example 1.8.1. Perform the indicated operations and simplify.

1. $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$
2. $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$
3. $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$
4. $\dfrac{\dfrac{2}{4-(x+h)} - \dfrac{2}{4-x}}{h}$
5. $2t^{-3}-(3t)^{-2}$
6. $10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2}$

Solution.

1. As with numeric fractions, we divide rational expressions by 'inverting and multiplying'. Before we get too carried away however, we factor to see what, if any, factors cancel.

$$\begin{aligned}
\frac{2x^2-5x-3}{x^4-4} \div \frac{x^2-2x-3}{x^5+2x^3} &= \frac{2x^2-5x-3}{x^4-4} \cdot \frac{x^5+2x^3}{x^2-2x-3} && \text{Invert and multiply} \\
&= \frac{(2x^2-5x-3)(x^5+2x^3)}{(x^4-4)(x^2-2x-3)} && \text{Multiply fractions} \\
&= \frac{(2x+1)(x-3)x^3(x^2+2)}{(x^2-2)(x^2+2)(x-3)(x+1)} && \text{Factor} \\
&= \frac{(2x+1)\cancel{(x-3)}x^3\cancel{(x^2+2)}}{(x^2-2)\cancel{(x^2+2)}\cancel{(x-3)}(x+1)} && \text{Cancel common factors} \\
&= \frac{x^3(2x+1)}{(x+1)(x^2-2)} && \text{Provided } x \neq 3
\end{aligned}$$

The '$x \neq 3$' is mentioned since a factor of $(x-3)$ was canceled as we reduced the expression. We also canceled a factor of (x^2+2). Why is there no stipulation as a result of canceling this factor? Because $x^2+2 \neq 0$. (Can you see why?) At this point, we *could* go ahead and multiply out the numerator and denominator to get

$$\frac{x^3(2x+1)}{(x+1)(x^2-2)} = \frac{2x^4+x^3}{x^3+x^2-2x-2}$$

but for most of the applications where this kind of algebra is needed (solving equations, for instance), it is best to leave things factored. Your instructor will let you know whether to leave your answer in factored form or not.[2]

[2]Speaking of factoring, do you remember why x^2-2 can't be factored over the integers?

2. As with numeric fractions we need common denominators in order to subtract. This is the case here so we proceed by subtracting the numerators.

$$\begin{aligned} \frac{5}{w^2-9} - \frac{w+2}{w^2-9} &= \frac{5-(w+2)}{w^2-9} && \text{Subtract fractions} \\ &= \frac{5-w-2}{w^2-9} && \text{Distribute} \\ &= \frac{3-w}{w^2-9} && \text{Combine like terms} \end{aligned}$$

At this point, we need to see if we can reduce this expression so we proceed to factor. It first appears as if we have no common factors among the numerator and denominator until we recall the property of 'factoring negatives' from Page 20: $3 - w = -(w - 3)$. This yields:

$$\begin{aligned} \frac{3-w}{w^2-9} &= \frac{-(w-3)}{(w-3)(w+3)} && \text{Factor} \\ &= \frac{-\cancel{(w-3)}}{\cancel{(w-3)}(w+3)} && \text{Cancel common factors} \\ &= \frac{-1}{w+3} && \text{Provided } w \neq 3 \end{aligned}$$

The stipulation $w \neq 3$ comes from the cancellation of the $(w - 3)$ factor.

3. In this next example, we are asked to add two rational expressions with *different* denominators. As with numeric fractions, we must first find a *common denominator*. To do so, we start by factoring each of the denominators.

$$\begin{aligned} \frac{3}{y^2-8y+16} + \frac{y+1}{16y-y^3} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(16-y^2)} && \text{Factor} \\ &= \frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} && \text{Factor some more} \end{aligned}$$

To find the common denominator, we examine the factors in the first denominator and note that we need a factor of $(y - 4)^2$. We now look at the second denominator to see what other factors we need. We need a factor of y and $(4 + y) = (y + 4)$. What about $(4 - y)$? As mentioned in the last example, we can factor this as: $(4 - y) = -(y - 4)$. Using properties of negatives, we 'migrate' this negative out to the front of the fraction, turning the addition into subtraction. We find the (least) common denominator to be $(y - 4)^2 y(y + 4)$. We can now proceed to multiply the numerator and denominator of each fraction by whatever factors each is missing from their respective denominators to produce equivalent expressions with common denominators.

$$\begin{aligned}
\frac{3}{(y-4)^2}+\frac{y+1}{y(4-y)(4+y)} &= \frac{3}{(y-4)^2}+\frac{y+1}{y(-(y-4))(y+4)} \\
&= \frac{3}{(y-4)^2}-\frac{y+1}{y(y-4)(y+4)} \\
&= \frac{3}{(y-4)^2}\cdot\frac{y(y+4)}{y(y+4)}-\frac{y+1}{y(y-4)(y+4)}\cdot\frac{(y-4)}{(y-4)} && \text{Equivalent Fractions} \\
&= \frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} && \text{Multiply Fractions}
\end{aligned}$$

At this stage, we can subtract numerators and simplify. We'll keep the denominator factored (in case we can reduce down later), but in the numerator, since there are no common factors, we proceed to perform the indicated multiplication and combine like terms.

$$\begin{aligned}
\frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} &= \frac{3y(y+4)-(y+1)(y-4)}{(y-4)^2y(y+4)} && \text{Subtract numerators} \\
&= \frac{3y^2+12y-(y^2-3y-4)}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{3y^2+12y-y^2+3y+4}{(y-4)^2y(y+4)} && \text{Distribute} \\
&= \frac{2y^2+15y+4}{y(y+4)(y-4)^2} && \text{Gather like terms}
\end{aligned}$$

We would like to factor the numerator and cancel factors it has in common with the denominator. After a few attempts, it appears as if the numerator doesn't factor, at least over the integers. As a check, we compute the discriminant of $2y^2+15y+4$ and get $15^2-4(2)(4)=193$. This isn't a perfect square so we know that the quadratic equation $2y^2+15y+4=0$ has irrational solutions. This means $2y^2+15y+4$ can't factor over the integers[3] so we are done.

4. In this example, we have a compound fraction, and we proceed to simplify it as we did its numeric counterparts in Example 1.2.1. Specifically, we start by multiplying the numerator and denominator of the 'big' fraction by the least common denominator of the 'little' fractions inside of it - in this case we need to use $(4-(x+h))(4-x)$ - to remove the compound nature of the 'big' fraction. Once we have a more normal looking fraction, we can proceed as we

[3]See the remarks following Theorem 1.10.

have in the previous examples.

$$\begin{aligned}
\frac{\frac{2}{4-(x+h)}-\frac{2}{4-x}}{h} &= \frac{\left(\frac{2}{4-(x+h)}-\frac{2}{4-x}\right)}{h}\cdot\frac{(4-(x+h))(4-x)}{(4-(x+h))(4-x)} && \text{Equivalent fractions}\\
&= \frac{\left(\frac{2}{4-(x+h)}-\frac{2}{4-x}\right)\cdot(4-(x+h))(4-x)}{h(4-(x+h))(4-x)} && \text{Multiply}\\
&= \frac{\frac{2(4-(x+h))(4-x)}{4-(x+h)}-\frac{2(4-(x+h))(4-x)}{4-x}}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{\frac{2\cancel{(4-(x+h))}(4-x)}{\cancel{(4-(x+h))}}-\frac{2(4-(x+h))\cancel{(4-x)}}{\cancel{(4-x)}}}{h(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)}
\end{aligned}$$

Now we can clean up and factor the numerator to see if anything cancels. (This why we kept the denominator factored.)

$$\begin{aligned}
\frac{2(4-x)-2(4-(x+h))}{h(4-(x+h))(4-x)} &= \frac{2[(4-x)-(4-(x+h))]}{h(4-(x+h))(4-x)} && \text{Factor out G.C.F.}\\
&= \frac{2[4-x-4+(x+h)]}{h(4-(x+h))(4-x)} && \text{Distribute}\\
&= \frac{2[4-4-x+x+h]}{h(4-(x+h))(4-x)} && \text{Rearrange terms}\\
&= \frac{2h}{h(4-(x+h))(4-x)} && \text{Gather like terms}\\
&= \frac{2\cancel{h}}{\cancel{h}(4-(x+h))(4-x)} && \text{Reduce}\\
&= \frac{2}{(4-(x+h))(4-x)} && \text{Provided } h \neq 0
\end{aligned}$$

Your instructor will let you know if you are to multiply out the denominator or not.[4]

5. At first glance, it doesn't seem as if there is anything that can be done with $2t^{-3}-(3t)^{-2}$ because the exponents on the variables are different. However, since the exponents are

[4] We'll keep it factored because in Calculus it needs to be factored.

negative, these are actually rational expressions. In the first term, the -3 exponent applies to the t *only* but in the second term, the exponent -2 applies to *both* the 3 and the t, as indicated by the parentheses. One way to proceed is as follows:

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= \frac{2}{t^3} - \frac{1}{(3t)^2} \\ &= \frac{2}{t^3} - \frac{1}{9t^2} \end{aligned}$$

We see that we are being asked to subtract two rational expressions with different denominators, so we need to find a common denominator. The first fraction contributes a t^3 to the denominator, while the second contributes a factor of 9. Thus our common denominator is $9t^3$, so we are missing a factor of '9' in the first denominator and a factor of 't' in the second.

$$\begin{aligned} \frac{2}{t^3} - \frac{1}{9t^2} &= \frac{2}{t^3} \cdot \frac{9}{9} - \frac{1}{9t^2} \cdot \frac{t}{t} && \text{Equivalent Fractions} \\ &= \frac{18}{9t^3} - \frac{t}{9t^3} && \text{Multiply} \\ &= \frac{18 - t}{9t^3} && \text{Subtract} \end{aligned}$$

We find no common factors among the numerator and denominator so we are done.

A second way to approach this problem is by factoring. We can extend the concept of the 'Polynomial G.C.F.' to these types of expressions and we can follow the same guidelines as set forth on page 71 to factor out the G.C.F. of these two terms. The key ideas to remember are that we take out each factor with the *smallest* exponent and factoring is the same as dividing. We first note that $2t^{-3} - (3t)^{-2} = 2t^{-3} - 3^{-2}t^{-2}$ and we see that the smallest power on t is -3. Thus we want to factor out t^{-3} from both terms. It's clear that this will leave 2 in the first term, but what about the second term? Since factoring is the same as dividing, we would be dividing the second term by t^{-3} which thanks to the properties of exponents is the same as *multiplying* by $\frac{1}{t^{-3}} = t^3$. The same holds for 3^{-2}. Even though there are no factors of 3 in the first term, we can factor out 3^{-2} by multiplying it by $\frac{1}{3^{-2}} = 3^2 = 9$. We put these ideas together below.

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= 2t^{-3} - 3^{-2}t^{-2} && \text{Properties of Exponents} \\ &= 3^{-2}t^{-3}(2(3)^2 - t^1) && \text{Factor} \\ &= \frac{1}{3^2}\frac{1}{t^3}(18 - t) && \text{Rewrite} \\ &= \frac{18 - t}{9t^3} && \text{Multiply} \end{aligned}$$

While both ways are valid, one may be more of a natural fit than the other depending on the circumstances and temperament of the student.

6. As with the previous example, we show two different yet equivalent ways to approach simplifying $10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2}$. First up is what we'll call the 'common denominator approach' where we rewrite the negative exponents as fractions and proceed from there.

   - *Common Denominator Approach*:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= \frac{10x}{x-3}+\frac{5x^2(-1)}{(x-3)^2} \\
&= \frac{10x}{x-3}\cdot\frac{x-3}{x-3}-\frac{5x^2}{(x-3)^2} && \text{Equivalent Fractions} \\
&= \frac{10x(x-3)}{(x-3)^2}-\frac{5x^2}{(x-3)^2} && \text{Multiply} \\
&= \frac{10x(x-3)-5x^2}{(x-3)^2} && \text{Subtract} \\
&= \frac{5x(2(x-3)-x)}{(x-3)^2} && \text{Factor out G.C.F.} \\
&= \frac{5x(2x-6-x)}{(x-3)^2} && \text{Distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Combine like terms}
\end{aligned}$$

   Both the numerator and the denominator are completely factored with no common factors so we are done.

   - *'Factoring Approach'*: In this case, the G.C.F. is $5x(x-3)^{-2}$. Factoring this out of both terms gives:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= 5x(x-3)^{-2}(2(x-3)^1-x) && \text{Factor} \\
&= \frac{5x}{(x-3)^2}(2x-6-x) && \text{Rewrite, distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Multiply}
\end{aligned}$$

   As expected, we got the same reduced fraction as before. □

Next, we review the solving of equations which involve rational expressions. As with equations involving numeric fractions, our first step in solving equations with algebraic fractions is to clear denominators. In doing so, we run the risk of introducing what are known as **extraneous** solutions - 'answers' which don't satisfy the original equation. As we illustrate the techniques used to solve these basic equations, see if you can find the step which creates the problem for us.

Example 1.8.2. Solve the following equations.

1. $1 + \dfrac{1}{x} = x$
2. $\dfrac{t^3 - 2t + 1}{t - 1} = \dfrac{1}{2}t - 1$
3. $\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w + 5} = 0$
4. $3(x^2 + 4)^{-1} + 3x(-1)(x^2 + 4)^{-2}(2x) = 0$
5. Solve $x = \dfrac{2y + 1}{y - 3}$ for y.
6. Solve $\dfrac{1}{f} = \dfrac{1}{S_1} + \dfrac{1}{S_2}$ for S_1.

Solution.

1. Our first step is to clear the fractions by multiplying both sides of the equation by x. In doing so, we are implicitly assuming $x \neq 0$; otherwise, we would have no guarantee that the resulting equation is equivalent to our original equation.[5]

$$\begin{array}{rcll}
1 + \dfrac{1}{x} & = & x & \\
\left(1 + \dfrac{1}{x}\right)x & = & (x)x & \text{Provided } x \neq 0 \\
1(x) + \dfrac{1}{x}(x) & = & x^2 & \text{Distribute} \\
x + \dfrac{x}{x} & = & x^2 & \text{Multiply} \\
x + 1 & = & x^2 & \\
0 & = & x^2 - x - 1 & \text{Subtract } x\text{, subtract } 1 \\
x & = & \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} & \text{Quadratic Formula} \\
x & = & \dfrac{1 \pm \sqrt{5}}{2} & \text{Simplify}
\end{array}$$

We obtain two answers, $x = \frac{1 \pm \sqrt{5}}{2}$. Neither of these are 0 thus neither contradicts our assumption that $x \neq 0$. The reader is invited to check both of these solutions.[6]

[5] See page 38.

[6] The check relies on being able to 'rationalize' the denominator - a skill we haven't reviewed yet. (Come back after you've read Section 1.9 if you want to!) Additionally, the positive solution to this equation is the famous Golden Ratio.

2. To solve the equation, we clear denominators. Here, we need to assume $t - 1 \neq 0$, or $t \neq 1$.

$$
\begin{array}{rcll}
\dfrac{t^3 - 2t + 1}{t - 1} & = & \dfrac{1}{2}t - 1 & \\
\left(\dfrac{t^3 - 2t + 1}{t - 1}\right) \cdot 2(t-1) & = & \left(\dfrac{1}{2}t - 1\right) \cdot 2(t-1) & \text{Provided } t \neq 1 \\
\dfrac{(t^3 - 2t + 1)(2\cancel{(t-1)})}{\cancel{(t-1)}} & = & \dfrac{1}{\cancel{2}}t(\cancel{2}(t-1)) - 1(2(t-1)) & \text{Multiply, distribute} \\
2(t^3 - 2t + 1) & = & t^2 - t - 2t + 2 & \text{Distribute} \\
2t^3 - 4t + 2 & = & t^2 - 3t + 2 & \text{Distribute, combine like terms} \\
2t^3 - t^2 - t & = & 0 & \text{Subtract } t^2\text{, add } 3t\text{, subtract } 2 \\
t(2t^2 - t - 1) & = & 0 & \text{Factor} \\
t = 0 & \text{or} & 2t^2 - t - 1 = 0 & \text{Zero Product Property} \\
t = 0 & \text{or} & (2t+1)(t-1) = 0 & \text{Factor} \\
t = 0 & \text{or} & 2t + 1 = 0 \quad \text{or} \quad t - 1 = 0 & \\
t & = & 0,\ -\dfrac{1}{2} \text{ or } 1 &
\end{array}
$$

We assumed that $t \neq 1$ in order to clear denominators. Sure enough, the 'solution' $t = 1$ doesn't check in the original equation since it causes division by 0. In this case, we call $t = 1$ an *extraneous* solution. Note that $t = 1$ *does* work in every equation *after* we clear denominators. In general, multiplying by variable expressions can produce these 'extra' solutions, which is why checking our answers is always encouraged.[7] The other two solutions, $t = 0$ and $t = -\frac{1}{2}$, both work.

3. As before, we begin by clearing denominators. Here, we assume $1 - w\sqrt{2} \neq 0$ (so $w \neq \frac{1}{\sqrt{2}}$) and $2w + 5 \neq 0$ (so $w \neq -\frac{5}{2}$).

$$
\begin{array}{rcll}
\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w+5} & = & 0 & \\
\left(\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w+5}\right)(1 - w\sqrt{2})(2w+5) & = & 0(1 - w\sqrt{2})(2w+5) & w \neq \dfrac{1}{\sqrt{2}}, -\dfrac{5}{2} \\
\dfrac{3\cancel{(1 - w\sqrt{2})}(2w+5)}{\cancel{(1 - w\sqrt{2})}} - \dfrac{1(1 - w\sqrt{2})\cancel{(2w+5)}}{\cancel{(2w+5)}} & = & 0 & \text{Distribute} \\
3(2w+5) - (1 - w\sqrt{2}) & = & 0 &
\end{array}
$$

The result is a *linear* equation in w so we gather the terms with w on one side of the equation

[7] Contrast this with what happened in Example 1.6.3 when we divided by a variable and 'lost' a solution.

and put everything else on the other. We factor out w and divide by its coefficient.

$$\begin{aligned}
3(2w+5)-(1-w\sqrt{2}) &= 0 & \\
6w+15-1+w\sqrt{2} &= 0 & \text{Distribute} \\
6w+w\sqrt{2} &= -14 & \text{Subtract 14} \\
(6+\sqrt{2})w &= -14 & \text{Factor} \\
w &= -\frac{14}{6+\sqrt{2}} & \text{Divide by } 6+\sqrt{2}
\end{aligned}$$

This solution is different than our excluded values, $\frac{1}{\sqrt{2}}$ and $-\frac{5}{2}$, so we keep $w=-\frac{14}{6+\sqrt{2}}$ as our final answer. The reader is invited to check this in the original equation.

4. To solve our next equation, we have two approaches to choose from: we could rewrite the quantities with negative exponents as fractions and clear denominators, or we can factor. We showcase each technique below.

   - *Clearing Denominators Approach*: We rewrite the negative exponents as fractions and clear denominators. In this case, we multiply both sides of the equation by $(x^2+4)^2$, which is never 0. (Think about that for a moment.) As a result, we need not exclude any x values from our solution set.

$$\begin{aligned}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) &= 0 & \\
\frac{3}{x^2+4}+\frac{3x(-1)(2x)}{(x^2+4)^2} &= 0 & \text{Rewrite} \\
\left(\frac{3}{x^2+4}-\frac{6x^2}{(x^2+4)^2}\right)(x^2+4)^2 &= 0(x^2+4)^2 & \text{Multiply} \\
\frac{3\cancel{(x^2+4)^2}^{(x^2+4)}}{\cancel{(x^2+4)}}-\frac{6x^2\cancel{(x^2+4)^2}}{\cancel{(x^2+4)^2}} &= 0 & \text{Distribute} \\
3(x^2+4)-6x^2 &= 0 & \\
3x^2+12-6x^2 &= 0 & \text{Distribute} \\
-3x^2 &= -12 & \text{Combine like terms, subtract 12} \\
x^2 &= 4 & \text{Divide by } -3 \\
x &= \pm\sqrt{4}=\pm 2 & \text{Extract square roots}
\end{aligned}$$

   We leave it to the reader to show both $x=-2$ and $x=2$ satisfy the original equation.

   - *Factoring Approach*: Since the equation is already set equal to 0, we're ready to factor. Following the guidelines presented in Example 1.8.1, we factor out $3(x^2+4)^{-2}$ from both terms and look to see if more factoring can be done:

$$\begin{array}{rcll} 3(x^2+4)^{-1} + 3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\ 3(x^2+4)^{-2}((x^2+4)^1 + x(-1)(2x)) & = & 0 & \text{Factor} \\ 3(x^2+4)^{-2}(x^2+4-2x^2) & = & 0 & \\ 3(x^2+4)^{-2}(4-x^2) & = & 0 & \text{Gather like terms} \\ 3(x^2+4)^{-2} = 0 & \text{or} & 4 - x^2 = 0 & \text{Zero Product Property} \\ \dfrac{3}{x^2+4} = 0 & \text{or} & 4 = x^2 & \end{array}$$

The first equation yields no solutions (Think about this for a moment.) while the second gives us $x = \pm\sqrt{4} = \pm 2$ as before.

5. We are asked to solve this equation for y so we begin by clearing fractions with the stipulation that $y - 3 \neq 0$ or $y \neq 3$. We are left with a linear equation in the variable y. To solve this, we gather the terms containing y on one side of the equation and everything else on the other. Next, we factor out the y and divide by its coefficient, which in this case turns out to be $x - 2$. In order to divide by $x - 2$, we stipulate $x - 2 \neq 0$ or, said differently, $x \neq 2$.

$$\begin{array}{rcll} x & = & \dfrac{2y+1}{y-3} & \\ x(y-3) & = & \left(\dfrac{2y+1}{y-3}\right)(y-3) & \text{Provided } y \neq 3 \\ xy - 3x & = & \dfrac{(2y+1)\cancel{(y-3)}}{\cancel{(y-3)}} & \text{Distribute, multiply} \\ xy - 3x & = & 2y + 1 & \\ xy - 2y & = & 3x + 1 & \text{Add } 3x\text{, subtract } 2y \\ y(x-2) & = & 3x + 1 & \text{Factor} \\ y & = & \dfrac{3x+1}{x-2} & \text{Divide provided } x \neq 2 \end{array}$$

We highly encourage the reader to check the answer algebraically to see where the restrictions on x and y come into play.[8]

6. Our last example comes from physics and the world of photography.[9] We take a moment here to note that while superscripts in mathematics indicate exponents (powers), subscripts are used primarily to distinguish one or more variables. In this case, S_1 and S_2 are two *different* variables (much like x and y) and we treat them as such. Our first step is to clear

[8] It involves simplifying a compound fraction!
[9] See this article on focal length.

denominators by multiplying both sides by fS_1S_2 - provided each is nonzero. We end up with an equation which is linear in S_1 so we proceed as in the previous example.

$$\begin{aligned}
\frac{1}{f} &= \frac{1}{S_1}+\frac{1}{S_2} \\
\left(\frac{1}{f}\right)(fS_1S_2) &= \left(\frac{1}{S_1}+\frac{1}{S_2}\right)(fS_1S_2) && \text{Provided } f\neq 0,\ S_1\neq 0,\ S_2\neq 0 \\
\frac{fS_1S_2}{f} &= \frac{fS_1S_2}{S_1}+\frac{fS_1S_2}{S_2} && \text{Multiply, distribute} \\
\frac{\cancel{f}S_1S_2}{\cancel{f}} &= \frac{f\cancel{S_1}S_2}{\cancel{S_1}}+\frac{fS_1\cancel{S_2}}{\cancel{S_2}} && \text{Cancel} \\
S_1S_2 &= fS_2+fS_1 \\
S_1S_2-fS_1 &= fS_2 && \text{Subtract } fS_1 \\
S_1(S_2-f) &= fS_2 && \text{Factor} \\
S_1 &= \frac{fS_2}{S_2-f} && \text{Divide provided } S_2\neq f
\end{aligned}$$

As always, the reader is highly encouraged to check the answer.[10] □

[10]. . . and see what the restriction $S_2 \neq f$ means in terms of focusing a camera!

1.8.1 Exercises

In Exercises 1 - 18, perform the indicated operations and simplify.

1. $\dfrac{x^2-9}{x^2} \cdot \dfrac{3x}{x^2-x-6}$

2. $\dfrac{t^2-2t}{t^2+1} \div (3t^2-2t-8)$

3. $\dfrac{4y-y^2}{2y+1} \div \dfrac{y^2-16}{2y^2-5y-3}$

4. $\dfrac{x}{3x-1} - \dfrac{1-x}{3x-1}$

5. $\dfrac{2}{w-1} - \dfrac{w^2+1}{w-1}$

6. $\dfrac{2-y}{3y} - \dfrac{1-y}{3y} + \dfrac{y^2-1}{3y}$

7. $b + \dfrac{1}{b-3} - 2$

8. $\dfrac{2x}{x-4} - \dfrac{1}{2x+1}$

9. $\dfrac{m^2}{m^2-4} + \dfrac{1}{2-m}$

10. $\dfrac{\frac{2}{x}-2}{x-1}$

11. $\dfrac{\frac{3}{2-h}-\frac{3}{2}}{h}$

12. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

13. $3w^{-1} - (3w)^{-1}$

14. $-2y^{-1} + 2(3-y)^{-2}$

15. $3(x-2)^{-1} - 3x(x-2)^{-2}$

16. $\dfrac{t^{-1}+t^{-2}}{t^{-3}}$

17. $\dfrac{2(3+h)^{-2}-2(3)^{-2}}{h}$

18. $\dfrac{(7-x-h)^{-1}-(7-x)^{-1}}{h}$

In Exercises 19 - 27, find all real solutions. Be sure to check for extraneous solutions.

19. $\dfrac{x}{5x+4} = 3$

20. $\dfrac{3y-1}{y^2+1} = 1$

21. $\dfrac{1}{w+3} + \dfrac{1}{w-3} = \dfrac{w^2-3}{w^2-9}$

22. $\dfrac{2x+17}{x+1} = x+5$

23. $\dfrac{t^2-2t+1}{t^3+t^2-2t} = 1$

24. $\dfrac{-y^3+4y}{y^2-9} = 4y$

25. $w + \sqrt{3} = \dfrac{3w-w^3}{w-\sqrt{3}}$

26. $\dfrac{2}{x\sqrt{2}-1} - 1 = \dfrac{3}{x\sqrt{2}+1}$

27. $\dfrac{x^2}{(1+x\sqrt{3})^2} = 3$

In Exercises 28 - 30, use Theorem 1.3 along with the techniques in this section to find all real solutions.

28. $\left|\dfrac{3n}{n-1}\right| = 3$

29. $\left|\dfrac{2x}{x^2-1}\right| = 2$

30. $\left|\dfrac{2t}{4-t^2}\right| = \left|\dfrac{2}{t-2}\right|$

In Exercises 31 - 33, find all real solutions and use a calculator to approximate your answers, rounded to two decimal places.

31. $2.41 = \dfrac{0.08}{4\pi R^2}$

32. $\dfrac{x^2}{(2.31-x)^2} = 0.04$

33. $1 - \dfrac{6.75 \times 10^{16}}{c^2} = \dfrac{1}{4}$

In Exercises 34 - 39, solve the given equation for the indicated variable.

34. Solve for y: $\dfrac{1-2y}{y+3} = x$

35. Solve for y: $x = 3 - \dfrac{2}{1-y}$

36.[11] Solve for T_2: $\dfrac{V_1}{T_1} = \dfrac{V_2}{T_2}$

37. Solve for t_0: $\dfrac{t_0}{1 - t_0 t_1} = 2$

38. Solve for x: $\dfrac{1}{x - v_r} + \dfrac{1}{x + v_r} = 5$

39. Solve for R: $P = \dfrac{25R}{(R+4)^2}$

[11] Recall: subscripts on variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat quantities like 'V_1' and 'V_2' as two different variables as you would 'x' and 'y.'

1.8.2 Answers

1. $\dfrac{3(x+3)}{x(x+2)}$, $x \neq 3$
2. $\dfrac{t}{(3t+4)(t^2+1)}$, $t \neq 2$
3. $-\dfrac{y(y-3)}{y+4}$, $y \neq -\dfrac{1}{2}, 3, 4$
4. $\dfrac{2x-1}{3x-1}$
5. $-w-1$, $w \neq 1$
6. $\dfrac{y}{3}$, $y \neq 0$
7. $\dfrac{b^2-5b+7}{b-3}$
8. $\dfrac{4x^2+x+4}{(x-4)(2x+1)}$
9. $\dfrac{m+1}{m+2}$, $m \neq 2$
10. $-\dfrac{2}{x}$, $x \neq 1$
11. $\dfrac{3}{4-2h}$, $h \neq 0$
12. $-\dfrac{1}{x(x+h)}$, $h \neq 0$
13. $\dfrac{8}{3w}$
14. $-\dfrac{2(y^2-7y+9)}{y(y-3)^2}$
15. $-\dfrac{6}{(x-2)^2}$
16. t^2+t, $t \neq 0$
17. $-\dfrac{2(h+6)}{9(h+3)^2}$, $h \neq 0$
18. $\dfrac{1}{(7-x)(7-x-h)}$, $h \neq 0$
19. $x = -\dfrac{6}{7}$
20. $y = 1, 2$
21. $w = -1$
22. $x = -6, 2$
23. No solution.
24. $y = 0, \pm 2\sqrt{2}$
25. $w = -\sqrt{3}, -1$
26. $x = -\dfrac{3\sqrt{2}}{2}, \sqrt{2}$
27. $x = -\dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{4}$
28. $n = \dfrac{1}{2}$
29. $x = \dfrac{1 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{5}}{2}$
30. $t = -1$
31. $R = \pm\sqrt{\dfrac{0.08}{9.64\pi}} \approx \pm 0.05$
32. $x = -\dfrac{231}{400} \approx -0.58$, $x = \dfrac{77}{200} \approx 0.38$
33. $c = \pm\sqrt{\dfrac{4 \cdot 6.75 \times 10^{16}}{3}} = \pm 3.00 \times 10^8$ (You actually didn't *need* a calculator for this!)
34. $y = \dfrac{1-3x}{x+2}$, $y \neq -3$, $x \neq -2$
35. $y = \dfrac{x-1}{x-3}$, $y \neq 1$, $x \neq 3$
36. $T_2 = \dfrac{V_2 T_1}{V_1}$, $T_1 \neq 0$, $T_2 \neq 0$, $V_1 \neq 0$
37. $t_0 = \dfrac{2}{2t_1+1}$, $t_1 \neq -\dfrac{1}{2}$
38. $x = \dfrac{1 \pm \sqrt{25v_r^2+1}}{5}$, $x \neq \pm v_r$.
39. $R = \dfrac{-(8P-25) \pm \sqrt{(8P-25)^2 - 64P^2}}{2P} = \dfrac{(25-8P) \pm 5\sqrt{25-16P}}{2P}$, $P \neq 0$, $R \neq -4$

1.9 Radicals and Equations

In this section we review simplifying expressions and solving equations involving radicals. In addition to the product, quotient and power rules stated in Theorem 1.1 in Section 1.2, we present the following result which states that n^{th} roots and n^{th} powers more or less 'undo' each other.[1]

Theorem 1.11. Simplifying n^{th} powers of n^{th} roots: Suppose n is a natural number, a is a real number and $\sqrt[n]{a}$ is a real number. Then

- $(\sqrt[n]{a})^n = a$
- if n is odd, $\sqrt[n]{a^n} = a$; if n is even, $\sqrt[n]{a^n} = |a|$.

Since $\sqrt[n]{a}$ is *defined* so that $(\sqrt[n]{a})^n = a$, the first claim in the theorem is just a re-wording of Definition 1.8. The second part of the theorem breaks down along odd/even exponent lines due to how exponents affect negatives. To see this, consider the specific cases of $\sqrt[3]{(-2)^3}$ and $\sqrt[4]{(-2)^4}$.

In the first case, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$, so we have an instance of when $\sqrt[n]{a^n} = a$. The reason that the cube root 'undoes' the third power in $\sqrt[3]{(-2)^3} = -2$ is because the negative is preserved when raised to the third (odd) power. In $\sqrt[4]{(-2)^4}$, the negative 'goes away' when raised to the fourth (even) power: $\sqrt[4]{(-2)^4} = \sqrt[4]{16}$. According to Definition 1.8, the fourth root is defined to give only *non-negative* numbers, so $\sqrt[4]{16} = 2$. Here we have a case where $\sqrt[4]{(-2)^4} = 2 = |-2|$, not -2.

In general, we need the absolute values to simplify $\sqrt[n]{a^n}$ only when n is even because a negative to an even power is always positive. In particular, $\sqrt{x^2} = |x|$, not just 'x' (unless we *know* $x \geq 0$.)[2] We practice these formulas in the following example.

Example 1.9.1. Perform the indicated operations and simplify.

1. $\sqrt{x^2+1}$
2. $\sqrt{t^2-10t+25}$
3. $\sqrt[3]{48x^{14}}$
4. $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$
5. $2x\sqrt[3]{x^2-4}+2\left(\dfrac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x)$
6. $\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2}$

Solution.

1. We told you back on page 32 that roots do not 'distribute' across addition and since x^2+1 cannot be factored over the real numbers, $\sqrt{x^2+1}$ cannot be simplified. It may seem silly to start with this example but it is extremely important that you understand what maneuvers are legal and which ones are not.[3]

[1] See Section **??** for a more precise understanding of what we mean here.

[2] If this discussion sounds familiar, see the discussion following Definition 1.9 and the discussion following 'Extracting the Square Root' on page 83.

[3] You really do need to understand this otherwise horrible evil will plague your future studies in Math. If you say something totally wrong like $\sqrt{x^2+1} = x+1$ then you may never pass Calculus. PLEASE be careful!

2. Again we note that $\sqrt{t^2 - 10t + 25} \neq \sqrt{t^2} - \sqrt{10t} + \sqrt{25}$, since radicals do *not* distribute across addition and subtraction.[4] In this case, however, we can factor the radicand and simplify as

$$\sqrt{t^2 - 10t + 25} = \sqrt{(t-5)^2} = |t-5|$$

Without knowing more about the value of t, we have no idea if $t - 5$ is positive or negative so $|t-5|$ is our final answer.[5]

3. To simplify $\sqrt[3]{48x^{14}}$, we need to look for perfect cubes in the radicand. For the cofficient, we have $48 = 8 \cdot 6 = 2^3 \cdot 6$. To find the largest perfect cube factor in x^{14}, we divide 14 (the exponent on x) by 3 (since we are looking for a perfect *cube*). We get 4 with a remainder of 2. This means $14 = 4 \cdot 3 + 2$, so $x^{14} = x^{4 \cdot 3 + 2} = x^{4 \cdot 3} x^2 = (x^4)^3 x^2$. Putting this altogether gives:

$$\begin{array}{rcll} \sqrt[3]{48x^{14}} & = & \sqrt[3]{2^3 \cdot 6 \cdot (x^4)^3 x^2} & \text{Factor out perfect cubes} \\ & = & \sqrt[3]{2^3} \sqrt[3]{(x^4)^3} \sqrt[3]{6x^2} & \text{Rearrange factors, Product Rule of Radicals} \\ & = & 2x^4 \sqrt[3]{6x^2} & \end{array}$$

4. In this example, we are looking for perfect fourth powers in the radicand. In the numerator r^4 is clearly a perfect fourth power. For the denominator, we take the power on the L, namely 12, and divide by 4 to get 3. This means $L^8 = L^{2 \cdot 4} = (L^2)^4$. We get

$$\begin{array}{rcll} \sqrt[4]{\dfrac{\pi r^4}{L^{12}}} & = & \dfrac{\sqrt[4]{\pi r^4}}{\sqrt[4]{L^{12}}} & \text{Quotient Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}\sqrt[4]{r^4}}{\sqrt[4]{(L^2)^4}} & \text{Product Rule of Radicals} \\ & = & \dfrac{\sqrt[4]{\pi}|r|}{|L^2|} & \text{Simplify} \end{array}$$

Without more information about r, we cannot simplify $|r|$ any further. However, we can simplify $|L^2|$. Regardless of the choice of L, $L^2 \geq 0$. Actually, $L^2 > 0$ because L is in the denominator which means $L \neq 0$. Hence, $|L^2| = L^2$. Our answer simplifies to:

$$\frac{\sqrt[4]{\pi}|r|}{|L^2|} = \frac{|r|\sqrt[4]{\pi}}{L^2}$$

5. After a quick cancellation (two of the 2's in the second term) we need to obtain a common denominator. Since we can view the first term as having a denominator of 1, the common denominator is precisely the denominator of the second term, namely $(\sqrt[3]{x^2 - 4})^2$. With

[4] Let $t = 1$ and see what happens to $\sqrt{t^2 - 10t + 25}$ versus $\sqrt{t^2} - \sqrt{10t} + \sqrt{25}$.

[5] In general, $|t-5| \neq |t| - |5|$ and $|t-5| \neq t + 5$ so watch what you're doing!

common denominators, we proceed to add the two fractions. Our last step is to factor the numerator to see if there are any cancellation opportunities with the denominator.

$$\begin{aligned}
2x\sqrt[3]{x^2-4}+2\left(\frac{1}{2(\sqrt[3]{x^2-4})^2}\right)(2x) &= 2x\sqrt[3]{x^2-4}+\cancel{2}\left(\frac{1}{\cancel{2}(\sqrt[3]{x^2-4})^2}\right)(2x) && \text{Reduce}\\
&= 2x\sqrt[3]{x^2-4}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Mutiply}\\
&= (2x\sqrt[3]{x^2-4})\cdot\frac{(\sqrt[3]{x^2-4})^2}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Equivalent fractions}\\
&= \frac{2x(\sqrt[3]{x^2-4})^3}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Multiply}\\
&= \frac{2x(x^2-4)}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Simplify}\\
&= \frac{2x(x^2-4)+2x}{(\sqrt[3]{x^2-4})^2} && \text{Add}\\
&= \frac{2x(x^2-4+1)}{(\sqrt[3]{x^2-4})^2} && \text{Factor}\\
&= \frac{2x(x^2-3)}{(\sqrt[3]{x^2-4})^2}
\end{aligned}$$

We cannot reduce this any further because x^2-3 is irreducible over the rational numbers.

6. We begin by working inside each set of parentheses, using the product rule for radicals and combining like terms.

$$\begin{aligned}
\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2} &= \sqrt{(\sqrt{9\cdot 2y}-\sqrt{4\cdot 2y})^2+(\sqrt{4\cdot 5}-\sqrt{16\cdot 5})^2}\\
&= \sqrt{(\sqrt{9}\sqrt{2y}-\sqrt{4}\sqrt{2y})^2+(\sqrt{4}\sqrt{5}-\sqrt{16}\sqrt{5})^2}\\
&= \sqrt{(3\sqrt{2y}-2\sqrt{2y})^2+(2\sqrt{5}-4\sqrt{5})^2}\\
&= \sqrt{(\sqrt{2y})^2+(-2\sqrt{5})^2}\\
&= \sqrt{2y+(-2)^2(\sqrt{5})^2}\\
&= \sqrt{2y+4\cdot 5}\\
&= \sqrt{2y+20}
\end{aligned}$$

To see if this simplifies any further, we factor the radicand: $\sqrt{2y+20}=\sqrt{2(y+10)}$. Finding no perfect square factors, we are done. □

Theorem 1.11 allows us to generalize the process of 'Extracting Square Roots' to 'Extracting n^{th} roots' which in turn allows us to solve equations[6] of the form $X^n = c$.

Extracting n^{th} roots:

- If c is a real number and n is odd then the real number solution to $X^n = c$ is $X = \sqrt[n]{c}$.
- If $c \geq 0$ and n is even then the real number solutions to $X^n = c$ are $X = \pm\sqrt[n]{c}$.

  Note: If $c < 0$ and n is even then $X^n = c$ has no real number solutions.

Essentially, we solve $X^n = c$ by 'taking the n^{th} root' of both sides: $\sqrt[n]{X^n} = \sqrt[n]{c}$. Simplifying the left side gives us just X if n is odd or $|X|$ if n is even. In the first case, $X = \sqrt[n]{c}$, and in the second, $X = \pm\sqrt[n]{c}$. Putting this together with the other part of Theorem 1.11, namely $(\sqrt[n]{a})^n = a$, gives us a strategy for solving equations which involve n^{th} and n^{th} roots.

Strategies for Power and Radical Equations

- If the equation involves an n^{th} power and the variable appears in only one term, isolate the term with the n^{th} power and extract n^{th} roots.
- If the equation involves an n^{th} root and the variable appears in that n^{th} root, isolate the n^{th} root and raise both sides of the equation to the n^{th} power.

  Note: When raising both sides of an equation to an *even* power, be sure to check for extraneous solutions.

The note about 'extraneous solutions' can be demonstrated by the basic equation: $\sqrt{x} = -2$. This equation has no solution since, by definition, $\sqrt{x} \geq 0$ for all real numbers x. However, if we square both sides of this equation, we get $(\sqrt{x})^2 = (-2)^2$ or $x = 4$. However, $x = 4$ doesn't check in the original equation, since $\sqrt{4} = 2$, not -2. Once again, the root[7] of all of our problems lies in the fact that a *negative* number to an *even* power results in a *positive* number. In other words, raising both sides of an equation to an even power does *not* produce an equivalent equation, but rather, an equation which may possess *more* solutions than the original. Hence the cautionary remark above about extraneous solutions.

Example 1.9.2. Solve the following equations.

1. $(5x+3)^4 = 16$
2. $1 - \dfrac{(5-2w)^3}{7} = 9$
3. $t + \sqrt{2t+3} = 6$
4. $\sqrt{2} - 3\sqrt[3]{2y+1} = 0$
5. $\sqrt{4x-1} + 2\sqrt{1-2x} = 1$
6. $\sqrt[4]{n^2+2} + n = 0$

For the remaining problems, assume that all of the variables represent positive real numbers.[8]

[6] Well, not entirely. The equation $x^7 = 1$ has seven answers: $x = 1$ and six complex number solutions which we'll find using techniques in Section **??**.

[7] Pun intended!

[8] That is, you needn't worry that you're multiplying or dividing by 0 or that you're forgetting absolute value symbols.

7. Solve for r: $V = \frac{4\pi}{3}(R^3 - r^3)$.

8. Solve for M_1: $\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$

9. Solve for v: $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$. Assume all quantities represent positive real numbers.

Solution.

1. In our first equation, the quantity containing x is already isolated, so we extract fourth roots. Since the exponent here is even, when the roots are extracted we need both the positive and negative roots.

$$\begin{aligned} (5x+3)^4 &= 16 & \\ 5x+3 &= \pm\sqrt[4]{16} & \text{Extract fourth roots} \\ 5x+3 &= \pm 2 & \\ 5x+3=2 \quad &\text{or} \quad 5x+3=-2 & \\ x = -\frac{1}{5} \quad &\text{or} \quad x=-1 & \end{aligned}$$

We leave it to the reader that both of these solutions satisfy the original equation.

2. In this example, we first need to isolate the quantity containing the variable w. Here, third (cube) roots are required and since the exponent (index) is odd, we do not need the $\pm$:

$$\begin{aligned} 1 - \frac{(5-2w)^3}{7} &= 9 & \\ -\frac{(5-2w)^3}{7} &= 8 & \text{Subtract } 1 \\ (5-2w)^3 &= -56 & \text{Multiply by } -7 \\ 5-2w &= \sqrt[3]{-56} & \text{Extract cube root} \\ 5-2w &= \sqrt[3]{(-8)(7)} & \\ 5-2w &= \sqrt[3]{-8}\sqrt[3]{7} & \text{Product Rule} \\ 5-2w &= -2\sqrt[3]{7} & \\ -2w &= -5-2\sqrt[3]{7} & \text{Subtract } 5 \\ w &= \frac{-5-2\sqrt[3]{7}}{-2} & \text{Divide by } -2 \\ w &= \frac{5+2\sqrt[3]{7}}{2} & \text{Properties of Negatives} \end{aligned}$$

The reader should check the answer because it provides a hearty review of arithmetic.

3. To solve $t + \sqrt{2t+3} = 6$, we first isolate the square root, then proceed to square both sides of the equation. In doing so, we run the risk of introducing extraneous solutions so checking

our answers here is a necessity.

$$
\begin{array}{rcll}
t+\sqrt{2t+3} & = & 6 & \\
\sqrt{2t+3} & = & 6-t & \text{Subtract } t \\
(\sqrt{2t+3})^2 & = & (6-t)^2 & \text{Square both sides} \\
2t+3 & = & 36-12t+t^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
0 & = & t^2-14t+33 & \text{Subtract } 2t \text{ and } 3 \\
0 & = & (t-3)(t-11) & \text{Factor}
\end{array}
$$

From the Zero Product Property, we know either $t-3=0$ (which gives $t=3$) or $t-11=0$ (which gives $t=11$). When checking our answers, we find $t=3$ satisfies the original equation, but $t=11$ does not.[9] So our final answer is $t=3$ only.

4. In our next example, we locate the variable (in this case y) beneath a cube root, so we first isolate that root and cube both sides.

$$
\begin{array}{rcll}
\sqrt{2}-3\sqrt[3]{2y+1} & = & 0 & \\
-3\sqrt[3]{2y+1} & = & -\sqrt{2} & \text{Subtract } \sqrt{2} \\
\sqrt[3]{2y+1} & = & \dfrac{-\sqrt{2}}{-3} & \text{Divide by } -3 \\
\sqrt[3]{2y+1} & = & \dfrac{\sqrt{2}}{3} & \text{Properties of Negatives} \\
\left(\sqrt[3]{2y+1}\right)^3 & = & \left(\dfrac{\sqrt{2}}{3}\right)^3 & \text{Cube both sides} \\
2y+1 & = & \dfrac{(\sqrt{2})^3}{3^3} & \\
2y+1 & = & \dfrac{2\sqrt{2}}{27} & \\
2y & = & \dfrac{2\sqrt{2}}{27}-1 & \text{Subtract 1} \\
2y & = & \dfrac{2\sqrt{2}}{27}-\dfrac{27}{27} & \text{Common denominators} \\
2y & = & \dfrac{2\sqrt{2}-27}{27} & \text{Subtract fractions} \\
y & = & \dfrac{2\sqrt{2}-27}{54} & \text{Divide by 2 } \left(\text{multiply by } \frac{1}{2}\right)
\end{array}
$$

Since we raised both sides to an *odd* power, we don't need to worry about extraneous solutions but we encourage the reader to check the solution just for the fun of it.

[9] It is worth noting that when $t=11$ is substituted into the original equation, we get $11+\sqrt{25}=6$. If the $+\sqrt{25}$ were $-\sqrt{25}$, the solution would check. Once again, when squaring both sides of an equation, we lose track of $\pm$, which is what lets extraneous solutions in the door.

5. In the equation $\sqrt{4x-1}+2\sqrt{1-2x}=1$, we have not one but two square roots. We begin by isolating one of the square roots and squaring both sides.

$$\begin{array}{rcll}
\sqrt{4x-1}+2\sqrt{1-2x} & = & 1 & \\
\sqrt{4x-1} & = & 1-2\sqrt{1-2x} & \text{Subtract } 2\sqrt{1-2x} \text{ from both sides} \\
(\sqrt{4x-1})^2 & = & (1-2\sqrt{1-2x})^2 & \text{Square both sides} \\
4x-1 & = & 1-4\sqrt{1-2x}+(2\sqrt{1-2x})^2 & \text{F.O.I.L. / Perfect Square Trinomial} \\
4x-1 & = & 1-4\sqrt{1-2x}+4(1-2x) & \\
4x-1 & = & 1-4\sqrt{1-2x}+4-8x & \text{Distribute} \\
4x-1 & = & 5-8x-4\sqrt{1-2x} & \text{Gather like terms}
\end{array}$$

At this point, we have just one square root so we proceed to isolate it and square both sides a second time.[10]

$$\begin{array}{rcll}
4x-1 & = & 5-8x-4\sqrt{1-2x} & \\
12x-6 & = & -4\sqrt{1-2x} & \text{Subtract 5, add } 8x \\
(12x-6)^2 & = & (-4\sqrt{1-2x})^2 & \text{Square both sides} \\
144x^2-144x+36 & = & 16(1-2x) & \\
144x^2-144x+36 & = & 16-32x & \\
144x^2-112x+20 & = & 0 & \text{Subtract 16, add } 32x \\
4(36x^2-28x+5) & = & 0 & \text{Factor} \\
4(2x-1)(18x-5) & = & 0 & \text{Factor some more}
\end{array}$$

From the Zero Product Property, we know either $2x-1=0$ or $18x-5=0$. The former gives $x=\frac{1}{2}$ while the latter gives us $x=\frac{5}{18}$. Since we squared both sides of the equation (twice!), we need to check for extraneous solutions. We find $x=\frac{5}{18}$ to be extraneous, so our only solution is $x=\frac{1}{2}$.

6. As usual, our first step in solving $\sqrt[4]{n^2+2}+n=0$ is to isolate the radical. We then proceed to raise both sides to the fourth power to eliminate the fourth root:

$$\begin{array}{rcll}
\sqrt[4]{n^2+2}+n & = & 0 & \\
\sqrt[4]{n^2+2} & = & -n & \text{Subtract } n \\
(\sqrt[4]{n^2+2})^4 & = & (-n)^4 & \text{Raise both sides to the } 4^{\text{th}} \text{ power} \\
n^2+2 & = & n^4 & \text{Properties of Negatives} \\
0 & = & n^4-n^2-2 & \text{Subtract } n^2 \text{ and 2} \\
0 & = & (n^2-2)(n^2+1) & \text{Factor - this is a `Quadratic in Disguise'}
\end{array}$$

At this point, the Zero Product Property gives either $n^2-2=0$ or $n^2+1=0$. From $n^2-2=0$, we get $n^2=2$, so $n=\pm\sqrt{2}$. From $n^2+1=0$, we get $n^2=-1$, which gives no real solutions.[11]

[10] To avoid complications with fractions, we'll forego dividing by the coefficient of $\sqrt{1-2x}$, namely -4. This is perfectly fine so long as we don't forget to square it when we square both sides of the equation.

[11] Why is that again?

Since we raised both sides to an even (the fourth) power, we need to check for extraneous solutions. We find that $n = -\sqrt{2}$ works but $n = \sqrt{2}$ is extraneous.

7. In this problem, we are asked to solve for r. While there are a lot of letters in this equation[12], r appears in only one term: r^3. Our strategy is to isolate r^3 then extract the cube root.

$$\begin{array}{rcll}
V & = & \frac{4\pi}{3}(R^3 - r^3) & \\
3V & = & 4\pi(R^3 - r^3) & \text{Multiply by 3 to clear fractions} \\
3V & = & 4\pi R^3 - 4\pi r^3 & \text{Distribute} \\
3V - 4\pi R^3 & = & -4\pi r^3 & \text{Subtract } 4\pi R^3 \\
\frac{3V - 4\pi R^3}{-4\pi} & = & r^3 & \text{Divide by } -4\pi \\
\frac{4\pi R^3 - 3V}{4\pi} & = & r^3 & \text{Properties of Negatives} \\
\sqrt[3]{\frac{4\pi R^3 - 3V}{4\pi}} & = & r & \text{Extract the cube root}
\end{array}$$

The check is, as always, left to the reader and highly encouraged.

8. The equation we are asked to solve in this example is from the world of Chemistry and is none other than Graham's Law of effusion. As was mentioned in Example 1.8.2, subscripts in Mathematics are used to distinguish between variables and have no arithmetic significance. In this example, r_1, r_2, M_1 and M_2 are as different as x, y, z and 117. Since we are asked to solve for M_1, we locate M_1 and see it is in a denominator in a square root. We eliminate the square root by squaring both sides and proceed from there.

$$\begin{array}{rcll}
\frac{r_1}{r_2} & = & \sqrt{\frac{M_2}{M_1}} & \\
\left(\frac{r_1}{r_2}\right)^2 & = & \left(\sqrt{\frac{M_2}{M_1}}\right)^2 & \text{Square both sides} \\
\frac{r_1^2}{r_2^2} & = & \frac{M_2}{M_1} & \\
r_1^2 M_1 & = & M_2 r_2^2 & \text{Multiply by } r_2^2 M_1 \text{ to clear fractions, assume } r_2, M_1 \neq 0 \\
M_1 & = & \frac{M_2 r_2^2}{r_1^2} & \text{Divide by } r_1^2\text{, assume } r_1 \neq 0
\end{array}$$

As the reader may expect, checking the answer amounts to a good exercise in simplifying rational and radical expressions. The fact that we are assuming all of the variables represent positive real numbers comes in to play, as well.

[12]including a Greek letter, no less!

9. Our last equation to solve comes from Einstein's Special Theory of Relativity and relates the mass of an object to its velocity as it moves.[13] We are asked to solve for v which is located in just one term, namely v^2, which happens to lie in a fraction underneath a square root which is itself a denominator. We have quite a lot of work ahead of us!

$$\begin{aligned}
m &= \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} && \\
m\sqrt{1-\frac{v^2}{c^2}} &= m_0 && \text{Multiply by } \sqrt{1-\frac{v^2}{c^2}} \text{ to clear fractions} \\
\left(m\sqrt{1-\frac{v^2}{c^2}}\right)^2 &= m_0^2 && \text{Square both sides} \\
m^2\left(1-\frac{v^2}{c^2}\right) &= m_0^2 && \text{Properties of Exponents} \\
m^2 - \frac{m^2v^2}{c^2} &= m_0^2 && \text{Distribute} \\
-\frac{m^2v^2}{c^2} &= m_0^2 - m^2 && \text{Subtract } m^2 \\
m^2v^2 &= -c^2(m_0^2 - m^2) && \text{Multiply by } -c^2 \ (c^2 \neq 0) \\
m^2v^2 &= -c^2m_0^2 + c^2m^2 && \text{Distribute} \\
v^2 &= \frac{c^2m^2 - c^2m_0^2}{m^2} && \text{Rearrange terms, divide by } m^2 \ (m^2 \neq 0) \\
v &= \sqrt{\frac{c^2m^2 - c^2m_0^2}{m^2}} && \text{Extract Square Roots, } v > 0 \text{ so no } \pm \\
v &= \frac{\sqrt{c^2(m^2 - m_0^2)}}{\sqrt{m^2}} && \text{Properties of Radicals, factor} \\
v &= \frac{|c|\sqrt{m^2 - m_0^2}}{|m|} && \\
v &= \frac{c\sqrt{m^2 - m_0^2}}{m} && c > 0 \text{ and } m > 0 \text{ so } |c| = c \text{ and } |m| = m
\end{aligned}$$

Checking the answer algebraically would earn the reader great honor and respect on the Algebra battlefield so it is highly recommended.

1.9.1 Rationalizing Denominators and Numerators

In Section 1.7, there were a few instances where we needed to 'rationalize' a denominator - that is, take a fraction with radical in the denominator and re-write it as an equivalent fraction without

[13] See this article on the Lorentz Factor.

a radical in the denominator. There are various reasons for wanting to do this,[14] but the most pressing reason is that rationalizing denominators - and numerators as well - gives us an opportunity for more practice with fractions and radicals. To help refresh your memory, we rationalize a denominator and then a numerator below:

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{7\sqrt[3]{4}}{3} = \frac{7\sqrt[3]{4}\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{7\sqrt[3]{8}}{3\sqrt[3]{2}} = \frac{7\cdot 2}{3\sqrt[3]{2}} = \frac{14}{3\sqrt[3]{2}}$$

In general, if the fraction contains either a single term numerator or denominator with an undesirable n^{th} root, we multiply the numerator and denominator by whatever is required to obtain a perfect n^{th} power in the radicand that we want to eliminate. If the fraction contains two terms the situation is somewhat more complicated. To see why, consider the fraction $\frac{3}{4-\sqrt{5}}$. Suppose we wanted to rid the denominator of the $\sqrt{5}$ term. We could try as above and multiply numerator and denominator by $\sqrt{5}$ but that just yields:

$$\frac{3}{4-\sqrt{5}} = \frac{3\sqrt{5}}{(4-\sqrt{5})\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-5}$$

We haven't removed $\sqrt{5}$ from the denominator - we've just shuffled it over to the other term in the denominator. As you may recall, the strategy here is to multiply both numerator and denominator by what's called the **conjugate**.

Definition 1.17. Congugate of a Square Root Expression: If a, b and c are real numbers with $c > 0$ then the quantities $(a+b\sqrt{c})$ and $(a-b\sqrt{c})$ are **conjugates** of one another.[a] Conjugates multiply according to the Difference of Squares Formula:

$$(a+b\sqrt{c})(a-b\sqrt{c}) = a^2 - (b\sqrt{c})^2 = a^2 - b^2c$$

[a] As are $(b\sqrt{c}-a)$ and $(b\sqrt{c}+a)$: $(b\sqrt{c}-a)(b\sqrt{c}+a) = b^2c - a^2$.

That is, to get the conjugate of a two-term expression involving a square root, you change the '$-$' to a '$+$,' or vice-versa. For example, the conjugate of $4-\sqrt{5}$ is $4+\sqrt{5}$, and when we multiply these two factors together, we get $(4-\sqrt{5})(4+\sqrt{5}) = 4^2 - (\sqrt{5})^2 = 16-5 = 11$. Hence, to eliminate the $\sqrt{5}$ from the denominator of our original fraction, we multiply both the numerator and denominator by the *conjugate* of $4-\sqrt{5}$:

$$\frac{3}{4-\sqrt{5}} = \frac{3(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{3(4+\sqrt{5})}{4^2-(\sqrt{5})^2} = \frac{3(4+\sqrt{5})}{16-5} = \frac{12+3\sqrt{5}}{11}$$

What if we had $\sqrt[3]{5}$ instead of $\sqrt{5}$? We could try multiplying $4-\sqrt[3]{5}$ by $4+\sqrt[3]{5}$ to get

$$(4-\sqrt[3]{5})(4+\sqrt[3]{5}) = 4^2 - (\sqrt[3]{5})^2 = 16 - \sqrt[3]{25},$$

[14] Before the advent of the handheld calculator, rationalizing denominators made it easier to get decimal approximations to fractions containing radicals. However, some (admittedly more abstract) applications remain today – one of which we'll explore in Section 1.10; one you'll see in Calculus.

which leaves us with a cube root. What we need to undo the cube root is a perfect cube, which means we look to the Difference of Cubes Formula for inspiration: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. If we take $a = 4$ and $b = \sqrt[3]{5}$, we multiply

$$(4 - \sqrt[3]{5})(4^2 + 4\sqrt[3]{5} + (\sqrt[3]{5})^2) = 4^3 + 4^2\sqrt[3]{5} + 4\sqrt[3]{5} - 4^2\sqrt[3]{5} - 4(\sqrt[3]{5})^2 - (\sqrt[3]{5})^3 = 64 - 5 = 59$$

So if we were charged with rationalizing the denominator of $\frac{3}{4-\sqrt[3]{5}}$, we'd have:

$$\frac{3}{4 - \sqrt[3]{5}} = \frac{3(4^2 + 4\sqrt[3]{5} + (\sqrt[3]{5})^2)}{(4 - \sqrt[3]{5})(4^2 + 4\sqrt[3]{5} + (\sqrt[3]{5})^2)} = \frac{48 + 12\sqrt[3]{5} + 3\sqrt[3]{25}}{59}$$

This sort of thing extends to n^{th} roots since $(a - b)$ is a factor of $a^n - b^n$ for all natural numbers n, but in practice, we'll stick with square roots with just a few cube roots thrown in for a challenge.[15]

Example 1.9.3. Rationalize the indicated numerator or denominator:

1. Rationalize the denominator: $\dfrac{3}{\sqrt[5]{24x^2}}$
2. Rationalize the numerator: $\dfrac{\sqrt{9+h}-3}{h}$

Solution.

1. We are asked to rationalize the denominator, which in this case contains a fifth root. That means we need to work to create fifth powers of each of the factors of the radicand. To do so, we first factor the radicand: $24x^2 = 8 \cdot 3 \cdot x^2 = 2^3 \cdot 3 \cdot x^2$. To obtain fifth powers, we need to multiply by $2^2 \cdot 3^4 \cdot x^3$ inside the radical.

$$\begin{aligned}
\frac{3}{\sqrt[6]{24x^2}} &= \frac{3}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2}\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}} && \text{Equivalent Fractions} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^3 \cdot 3 \cdot x^2 \cdot 2^2 \cdot 3^4 \cdot x^3}} && \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5 \cdot 3^5 \cdot x^5}} && \text{Property of Exponents} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{\sqrt[5]{2^5}\sqrt[5]{3^5}\sqrt[5]{x^5}} && \text{Product Rule} \\
&= \frac{3\sqrt[5]{2^2 \cdot 3^4 \cdot x^3}}{2 \cdot 3 \cdot x} && \text{Product Rule} \\
&= \frac{\cancel{3}\sqrt[5]{4 \cdot 81 \cdot x^3}}{2 \cdot \cancel{3} \cdot x} && \text{Reduce} \\
&= \frac{\sqrt[5]{324x^3}}{2x} && \text{Simplify}
\end{aligned}$$

[15] To see what to do about fourth roots, use long division to find $(a^4 - b^4) \div (a - b)$, and apply this to $4 - \sqrt[4]{5}$.

2. Here, we are asked to rationalize the *numerator*. Since it is a two term numerator involving a square root, we multiply both numerator and denominator by the conjugate of $\sqrt{9+h}-3$, namely $\sqrt{9+h}+3$. After simplifying, we find an opportunity to reduce the fraction:

$$\begin{aligned}
\frac{\sqrt{9+h}-3}{h} &= \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} && \text{Equivalent Fractions}\\
&= \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} && \text{Difference of Squares}\\
&= \frac{(9+h)-9}{h(\sqrt{9+h}+3)} && \text{Simplify}\\
&= \frac{h}{h(\sqrt{9+h}+3)} && \text{Simplify}\\
&= \frac{\cancel{h}^{1}}{\cancel{h}(\sqrt{9+h}+3)} && \text{Reduce}\\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

We close this section with an awesome example from Calculus.

Example 1.9.4. Simplify the compound fraction $\dfrac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h}$ then rationalize the numerator of the result.

Solution. We start by multiplying the top and bottom of the 'big' fraction by $\sqrt{2x+2h+1}\sqrt{2x+1}$.

$$\begin{aligned}
\frac{\dfrac{1}{\sqrt{2(x+h)+1}}-\dfrac{1}{\sqrt{2x+1}}}{h} &= \frac{\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}}{h}\\
&= \frac{\left(\dfrac{1}{\sqrt{2x+2h+1}}-\dfrac{1}{\sqrt{2x+1}}\right)\sqrt{2x+2h+1}\sqrt{2x+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}\\
&= \frac{\dfrac{\cancel{\sqrt{2x+2h+1}}\sqrt{2x+1}}{\cancel{\sqrt{2x+2h+1}}}-\dfrac{\sqrt{2x+2h+1}\cancel{\sqrt{2x+1}}}{\cancel{\sqrt{2x+1}}}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}\\
&= \frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}
\end{aligned}$$

Next, we multiply the numerator and denominator by the conjugate of $\sqrt{2x+1}-\sqrt{2x+2h+1}$, namely $\sqrt{2x+1}+\sqrt{2x+2h+1}$, simplify and reduce:

$$\begin{aligned}
\frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} &= \frac{(\sqrt{2x+1}-\sqrt{2x+2h+1})(\sqrt{2x+1}+\sqrt{2x+2h+1})}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{(\sqrt{2x+1})^2-(\sqrt{2x+2h+1})^2}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{(2x+1)-(2x+2h+1)}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{2x+1-2x-2h-1}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{-2\cancel{h}}{\cancel{h}\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
&= \frac{-2}{\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})}
\end{aligned}$$

While the denominator is quite a bit more complicated than what we started with, we have done what was asked of us. In the interest of full disclosure, the reason we did all of this was to cancel the original 'h' from the denominator. That's an awful lot of effort to get rid of just one little h, but you'll see the significance of this in Calculus. □

1.9.2 Exercises

In Exercises 1 - 13, perform the indicated operations and simplify.

1. $\sqrt{9x^2}$
2. $\sqrt[3]{8t^3}$
3. $\sqrt{50y^6}$
4. $\sqrt{4t^2+4t+1}$
5. $\sqrt{w^2-16w+64}$
6. $\sqrt{(\sqrt{12x}-\sqrt{3x})^2+1}$
7. $\sqrt{\dfrac{c^2-v^2}{c^2}}$
8. $\sqrt[3]{\dfrac{24\pi r^5}{L^3}}$
9. $\sqrt[4]{\dfrac{32\pi\varepsilon^8}{\rho^{12}}}$
10. $\sqrt{x}-\dfrac{x+1}{\sqrt{x}}$
11. $3\sqrt{1-t^2}+3t\left(\dfrac{1}{2\sqrt{1-t^2}}\right)(-2t)$
12. $2\sqrt[3]{1-z}+2z\left(\dfrac{1}{3\left(\sqrt[3]{1-z}\right)^2}\right)(-1)$
13. $\dfrac{3}{\sqrt[3]{2x-1}}+(3x)\left(-\dfrac{1}{3\left(\sqrt[3]{2x-1}\right)^4}\right)(2)$

In Exercises 14 - 25, find all real solutions.

14. $(2x+1)^3+8=0$
15. $\dfrac{(1-2y)^4}{3}=27$
16. $\dfrac{1}{1+2t^3}=4$
17. $\sqrt{3x+1}=4$
18. $5-\sqrt[3]{t^2+1}=1$
19. $x+1=\sqrt{3x+7}$
20. $y+\sqrt{3y+10}=-2$
21. $3t+\sqrt{6-9t}=2$
22. $2x-1=\sqrt{x+3}$
23. $w=\sqrt[4]{12-w^2}$
24. $\sqrt{x-2}+\sqrt{x-5}=3$
25. $\sqrt{2x+1}=3+\sqrt{4-x}$

In Exercises 26 - 29, solve each equation for the indicated variable. Assume all quantities represent positive real numbers.

26. Solve for h: $I=\dfrac{bh^3}{12}$.
27. Solve for a: $I_0=\dfrac{5\sqrt{3}a^4}{16}$
28. Solve for g: $T=2\pi\sqrt{\dfrac{L}{g}}$
29. Solve for v: $L=L_0\sqrt{1-\dfrac{v^2}{c^2}}$.

In Exercises 30 - 35, rationalize the numerator or denominator, and simplify.

30. $\dfrac{4}{3-\sqrt{2}}$
31. $\dfrac{7}{\sqrt[3]{12x^7}}$
32. $\dfrac{\sqrt{x}-\sqrt{c}}{x-c}$
33. $\dfrac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h}$
34. $\dfrac{\sqrt[3]{x+1}-2}{x-7}$
35. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$

1.9.3 Answers

1. $3|x|$
2. $2t$
3. $5|y^3|\sqrt{2}$
4. $|2t+1|$
5. $|w-8|$
6. $\sqrt{3x+1}$
7. $\dfrac{\sqrt{c^2-v^2}}{|c|}$
8. $\dfrac{2r\sqrt[3]{3\pi r^2}}{L}$
9. $\dfrac{2\varepsilon^2\sqrt[4]{2\pi}}{|\rho^3|}$
10. $-\dfrac{1}{\sqrt{x}}$
11. $\dfrac{3-6t^2}{\sqrt{1-t^2}}$
12. $\dfrac{6-8z}{3(\sqrt[3]{1-z})^2}$
13. $\dfrac{4x-3}{(2x-1)\sqrt[3]{2x-1}}$
14. $x=-\dfrac{3}{2}$
15. $y=-1,2$
16. $t=-\dfrac{\sqrt[3]{3}}{2}$
17. $x=5$
18. $t=\pm 3\sqrt{7}$
19. $x=3$
20. $y=-3$
21. $t=-\dfrac{1}{3},\dfrac{2}{3}$
22. $x=\dfrac{5+\sqrt{57}}{8}$
23. $w=\sqrt{3}$
24. $x=6$
25. $x=4$
26. $h=\sqrt[3]{\dfrac{12I}{b}}$
27. $a=\dfrac{2\sqrt[4]{I_0}}{\sqrt[4]{5\sqrt{3}}}$
28. $g=\dfrac{4\pi^2 L}{T^2}$
29. $v=\dfrac{c\sqrt{L_0^2-L^2}}{L_0}$
30. $\dfrac{12+4\sqrt{2}}{7}$
31. $\dfrac{7\sqrt[3]{18x^2}}{6x^3}$
32. $\dfrac{1}{\sqrt{x}+\sqrt{c}}$
33. $\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$
34. $\dfrac{1}{(\sqrt[3]{x+1})^2+2\sqrt[3]{x+1}+4}$
35. $\dfrac{1}{(\sqrt[3]{x+h})^2+\sqrt[3]{x+h}\sqrt[3]{x}+(\sqrt[3]{x})^2}$

1.10 Complex Numbers

We conclude our Prerequisites chapter with a review the set of **Complex Numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number x with $x^2 = -1$, since for any real number $x^2 \geq 0$. However, we could formally extract square roots and write $x = \pm\sqrt{-1}$. We build the complex numbers by relabeling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**.[1] The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2 - 7i) + (3 + 4i) = 5 - 3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition 1.18. The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Property 1 in Definition 1.18 establishes that i does act as a square root[2] of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 1.18 in place, we are now in position to define the **complex numbers**.

Definition 1.19. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

Complex numbers include things you'd normally expect, like $3 + 2i$ and $\frac{2}{5} - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers,[3] so 0 and $\pi - \sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 1.18. The next example should help recall how these animals behave.

[1] Some Technical Mathematics textbooks label it 'j'. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

[2] Note the use of the indefinite article 'a'. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

[3] To use the language of Section 1.1.2, $\mathbb{R} \subseteq \mathbb{C}$.

Example 1.10.1. Perform the indicated operations.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

Solution.

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We distribute and combine like terms:

$$\begin{array}{rcll}(1-2i)-(3+4i) & = & 1-2i-3-4i & \text{Distribute} \\ & = & -2-6i & \text{Gather like terms}\end{array}$$

Technically, we'd have to rewrite our answer $-2-6i$ as $(-2)+(-6)i$ to be (in the strictest sense) 'in the form $a+bi$'. That being said, even pedants have their limits, and we'll consider $-2-6i$ good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{array}{rcll}(1-2i)(3+4i) & = & (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i) & \text{F.O.I.L.} \\ & = & 3+4i-6i-8i^2 & \\ & = & 3-2i-8(-1) & i^2=-1 \\ & = & 3-2i+8 & \\ & = & 11-2i & \end{array}$$

3. How in the world are we supposed to simplify $\frac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing two terms, one of which is a square root: we and multiply both numerator and denominator by $3+4i$, the (complex) conjugate of $3-4i$. Doing so produces

$$\begin{array}{rcll}\dfrac{1-2i}{3-4i} & = & \dfrac{(1-2i)(3+4i)}{(3-4i)(3+4i)} & \text{Equivalent Fractions} \\ & = & \dfrac{3+4i-6i-8i^2}{9-16i^2} & \text{F.O.I.L.} \\ & = & \dfrac{3-2i-8(-1)}{9-16(-1)} & i^2=-1 \\ & = & \dfrac{11-2i}{25} & \\ & = & \dfrac{11}{25}-\dfrac{2}{25}\,i & \end{array}$$

4. We use property 2 of Definition 1.18 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12}=\left(i\sqrt{3}\right)\left(i\sqrt{12}\right)=i^2\sqrt{3\cdot 12}=-\sqrt{36}=-6$.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}
(x - [1 + 2i])(x - [1 - 2i]) &= x^2 - x[1 - 2i] - x[1 + 2i] + [1 - 2i][1 + 2i] && \text{F.O.I.L.} \\
&= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 && \text{Distribute} \\
&= x^2 - 2x + 1 - 4(-1) && \text{Gather like terms} \\
&= x^2 - 2x + 5 && i^2 = -1
\end{aligned}$$

This type of factoring will be revisited in Section **??**. □

In the previous example, we used the 'conjugate' idea from Section 1.9 to divide two complex numbers. More generally, the **complex conjugate** of a complex number $a + bi$ is the number $a - bi$. The notation commonly used for complex conjugation is a 'bar': $\overline{a + bi} = a - bi$. For example, $\overline{3 + 2i} = 3 - 2i$ and $\overline{3 - 2i} = 3 + 2i$. To find $\overline{6}$, we note that $\overline{6} = \overline{6 + 0i} = 6 - 0i = 6$, so $\overline{6} = 6$. Similarly, $\overline{4i} = -4i$, since $\overline{4i} = \overline{0 + 4i} = 0 - 4i = -4i$. Note that $\overline{3 + \sqrt{5}} = 3 + \sqrt{5}$, not $3 - \sqrt{5}$, since $\overline{3 + \sqrt{5}} = \overline{3 + \sqrt{5} + 0i} = 3 + \sqrt{5} - 0i = 3 + \sqrt{5}$. Here, the conjugation specified by the 'bar' notation involves reversing the sign before $i = \sqrt{-1}$, not before $\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem 1.12. Properties of the Complex Conjugate: Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{z^n} = (\overline{z})^n$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Essentially, Theorem 1.12 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out $z = a + bi$ and $w = c + di$ for real numbers a, b, c and d. Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise.[4] To prove the second property, we compare $\overline{z + w}$ with $\overline{z} + \overline{w}$. We have $\overline{z} + \overline{w} = \overline{a + bi} + \overline{c + di} = a - bi + c - di$. To find $\overline{z + w}$, we first compute

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

so

$$\overline{z + w} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = a + c - bi - di = a - bi + c - di = \overline{z} + \overline{w}$$

[4] Trust us on this.

As such, we have established $\overline{z+w} = \overline{z} + \overline{w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction.[5] The last property is a characterization of real numbers. If z is real, then $z = a + 0i$, so $\overline{z} = a - 0i = a = z$. On the other hand, if $z = \overline{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

We now return to the business of solving quadratic equations. Consider $x^2 - 2x + 5 = 0$. The discriminant $b^2 - 4ac = -16$ is negative, so we know by Theorem 1.10 there are no *real* solutions, since the Quadratic Formula would involve the term $\sqrt{-16}$. Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Example 1.10.2. Find the complex solutions to the following equations.[6]

1. $\dfrac{2x}{x+1} = x + 3$
2. $2t^4 = 9t^2 + 5$
3. $z^3 + 1 = 0$

Solution.

1. Clearing fractions yields a quadratic equation so we then proceed as in Section 1.7.

$$\begin{array}{rcll}
\dfrac{2x}{x+1} & = & x+3 & \\
2x & = & (x+3)(x+1) & \text{Multiply by } (x+1) \text{ to clear denominators} \\
2x & = & x^2 + x + 3x + 3 & \text{F.O.I.L.} \\
2x & = & x^2 + 4x + 3 & \text{Gather like terms} \\
0 & = & x^2 + 2x + 3 & \text{Subtract } 2x
\end{array}$$

From here, we apply the Quadratic Formula

$$\begin{array}{rcll}
x & = & \dfrac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} & \text{Quadratic Formula} \\
& = & \dfrac{-2 \pm \sqrt{-8}}{2} & \text{Simplify} \\
& = & \dfrac{-2 \pm i\sqrt{8}}{2} & \text{Definition of } i \\
& = & \dfrac{-2 \pm i2\sqrt{2}}{2} & \text{Product Rule for Radicals} \\
& = & \dfrac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} & \text{Factor and reduce} \\
& = & -1 \pm i\sqrt{2} &
\end{array}$$

[5]See Section **??**.

[6]Remember, all real numbers are complex numbers, so ‘complex solutions’ means both real and non-real answers.

We get two answers: $x = -1 + i\sqrt{2}$ and its conjugate $x = -1 - i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2), we have a Quadratic in Disguise. We proceed accordingly.

$$\begin{array}{rcll} 2t^4 & = & 9t^2+5 & \\ 2t^4-9t^2-5 & = & 0 & \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2+1)(t^2-5) & = & 0 & \text{Factor} \\ 2t^2+1=0 & \text{or} & t^2=5 & \text{Zero Product Property} \end{array}$$

From $2t^2+1=0$ we get $2t^2=-1$, or $t^2=-\frac{1}{2}$. We extract square roots as follows:

$$t = \pm\sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2=5$, we get $t=\pm\sqrt{5}$. In total, we have four complex solutions - two real: $t=\pm\sqrt{5}$ and two non-real: $t=\pm\frac{i\sqrt{2}}{2}$.

3. To find the *real* solutions to $z^3+1=0$, we can subtract the 1 from both sides and extract cube roots: $z^3=-1$, so $z=\sqrt[3]{-1}=-1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{array}{rcll} z^3+1 & = & 0 & \\ (z+1)(z^2-z+1) & = & 0 & \text{Factor (Sum of Two Cubes)} \\ z+1=0 & \text{or} & z^2-z+1=0 & \end{array}$$

From $z+1=0$, we get our real solution $z=-1$. From $z^2-z+1=0$, we apply the Quadratic Formula to get:

$$z = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)} = \frac{1\pm\sqrt{-3}}{2} = \frac{1\pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to $z^3+1=0$ - one real: $z=-1$ and two non-real: $z=\frac{1\pm i\sqrt{3}}{2}$. As always, the reader is encouraged to test their algebraic mettle and check these solutions. □

It is no coincidence that the non-real solutions to the equations in Example 1.10.2 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem 1.10 which we state on the next page.

Theorem 1.13. Discriminant Theorem: Given a Quadratic Equation $AX^2 + BX + C = 0$, where A, B and C are real numbers, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one (repeated) real number solution.
  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B \pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section **??** and we will revisit Theorem 1.13 then.

1.10.1 Exercises

In Exercises 1 - 10, use the given complex numbers z and w to find and simplify the following.

- $z + w$
- zw
- z^2
- $\dfrac{1}{z}$
- $\dfrac{z}{w}$
- $\dfrac{w}{z}$
- $\overline{z}$
- $z\overline{z}$
- $(\overline{z})^2$

1. $z = 2 + 3i$, $w = 4i$
2. $z = 1 + i$, $w = -i$
3. $z = i$, $w = -1 + 2i$
4. $z = 4i$, $w = 2 - 2i$
5. $z = 3 - 5i$, $w = 2 + 7i$
6. $z = -5 + i$, $w = 4 + 2i$
7. $z = \sqrt{2} - i\sqrt{2}$, $w = \sqrt{2} + i\sqrt{2}$
8. $z = 1 - i\sqrt{3}$, $w = -1 - i\sqrt{3}$
9. $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$, $w = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$
10. $z = -\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$, $w = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$

In Exercises 11 - 18, simplify the quantity.

11. $\sqrt{-49}$
12. $\sqrt{-9}$
13. $\sqrt{-25}\sqrt{-4}$
14. $\sqrt{(-25)(-4)}$
15. $\sqrt{-9}\sqrt{-16}$
16. $\sqrt{(-9)(-16)}$
17. $\sqrt{-(-9)}$
18. $-\sqrt{(-9)}$

We know that $i^2 = -1$ which means $i^3 = i^2 \cdot i = (-1) \cdot i = -i$ and $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$. In Exercises 19 - 26, use this information to simplify the given power of i.

19. i^5
20. i^6
21. i^7
22. i^8
23. i^{15}
24. i^{26}
25. i^{117}
26. i^{304}

In Exercises 27 - 35, find all complex solutions.

27. $3x^2 + 6 = 4x$
28. $15t^2 + 2t + 5 = 3t(t^2 + 1)$
29. $3y^2 + 4 = y^4$
30. $\dfrac{2}{1 - w} = w$
31. $\dfrac{y}{3} - \dfrac{3}{y} = y$
32. $\dfrac{x^3}{2x - 1} = \dfrac{x}{3}$
33. $x = \dfrac{2}{\sqrt{5} - x}$
34. $\dfrac{5y^4 + 1}{y^2 - 1} = 3y^2$
35. $z^4 = 16$
36. Multiply and simplify: $\left(x - [3 - i\sqrt{23}]\right)\left(x - [3 + i\sqrt{23}]\right)$

1.10.2 Answers

1. For $z = 2 + 3i$ and $w = 4i$

   - $z + w = 2 + 7i$
   - $zw = -12 + 8i$
   - $z^2 = -5 + 12i$
   - $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}i$
   - $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}i$
   - $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}i$
   - $\overline{z} = 2 - 3i$
   - $z\overline{z} = 13$
   - $(\overline{z})^2 = -5 - 12i$

2. For $z = 1 + i$ and $w = -i$

   - $z + w = 1$
   - $zw = 1 - i$
   - $z^2 = 2i$
   - $\frac{1}{z} = \frac{1}{2} - \frac{1}{2}i$
   - $\frac{z}{w} = -1 + i$
   - $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}i$
   - $\overline{z} = 1 - i$
   - $z\overline{z} = 2$
   - $(\overline{z})^2 = -2i$

3. For $z = i$ and $w = -1 + 2i$

   - $z + w = -1 + 3i$
   - $zw = -2 - i$
   - $z^2 = -1$
   - $\frac{1}{z} = -i$
   - $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}i$
   - $\frac{w}{z} = 2 + i$
   - $\overline{z} = -i$
   - $z\overline{z} = 1$
   - $(\overline{z})^2 = -1$

4. For $z = 4i$ and $w = 2 - 2i$

   - $z + w = 2 + 2i$
   - $zw = 8 + 8i$
   - $z^2 = -16$
   - $\frac{1}{z} = -\frac{1}{4}i$
   - $\frac{z}{w} = -1 + i$
   - $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}i$
   - $\overline{z} = -4i$
   - $z\overline{z} = 16$
   - $(\overline{z})^2 = -16$

5. For $z = 3 - 5i$ and $w = 2 + 7i$

   - $z + w = 5 + 2i$
   - $zw = 41 + 11i$
   - $z^2 = -16 - 30i$
   - $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}i$
   - $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}i$
   - $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}i$
   - $\overline{z} = 3 + 5i$
   - $z\overline{z} = 34$
   - $(\overline{z})^2 = -16 + 30i$

6. For $z = -5 + i$ and $w = 4 + 2i$

- $z + w = -1 + 3i$
- $zw = -22 - 6i$
- $z^2 = 24 - 10i$
- $\frac{1}{z} = -\frac{5}{26} - \frac{1}{26}i$
- $\frac{z}{w} = -\frac{9}{10} + \frac{7}{10}i$
- $\frac{w}{z} = -\frac{9}{13} - \frac{7}{13}i$
- $\overline{z} = -5 - i$
- $z\overline{z} = 26$
- $(\overline{z})^2 = 24 + 10i$

7. For $z = \sqrt{2} - i\sqrt{2}$ and $w = \sqrt{2} + i\sqrt{2}$

- $z + w = 2\sqrt{2}$
- $zw = 4$
- $z^2 = -4i$
- $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\overline{z} = \sqrt{2} + i\sqrt{2}$
- $z\overline{z} = 4$
- $(\overline{z})^2 = 4i$

8. For $z = 1 - i\sqrt{3}$ and $w = -1 - i\sqrt{3}$

- $z + w = -2i\sqrt{3}$
- $zw = -4$
- $z^2 = -2 - 2i\sqrt{3}$
- $\frac{1}{z} = \frac{1}{4} + \frac{\sqrt{3}}{4}i$
- $\frac{z}{w} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\frac{w}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\overline{z} = 1 + i\sqrt{3}$
- $z\overline{z} = 4$
- $(\overline{z})^2 = -2 + 2i\sqrt{3}$

9. For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

- $z + w = i\sqrt{3}$
- $zw = -1$
- $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\overline{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $z\overline{z} = 1$
- $(\overline{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

10. For $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

- $z + w = -\sqrt{2}$
- $zw = 1$
- $z^2 = -i$
- $\frac{1}{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\overline{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$
- $z\overline{z} = 1$
- $(\overline{z})^2 = i$

11. $7i$

12. $3i$

13. -10

14. 10

15. -12

16. 12

17. 3

18. $-3i$

19. $i^5 = i^4 \cdot i = 1 \cdot i = i$

20. $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

21. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

22. $i^8 = i^4 \cdot i^4 = \left(i^4\right)^2 = (1)^2 = 1$

23. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

24. $i^{26} = \left(i^4\right)^6 \cdot i^2 = 1 \cdot (-1) = -1$

25. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

26. $i^{304} = \left(i^4\right)^{76} = 1^{76} = 1$

27. $x = \dfrac{2 \pm i\sqrt{14}}{3}$

28. $t = 5, \pm\dfrac{i\sqrt{3}}{3}$

29. $y = \pm 2, \pm i$

30. $w = \dfrac{1 \pm i\sqrt{7}}{2}$

31. $y = \pm\dfrac{3i\sqrt{2}}{2}$

32. $x = 0, \dfrac{1 \pm i\sqrt{2}}{3}$

33. $x = \dfrac{\sqrt{5} \pm i\sqrt{3}}{2}$

34. $y = \pm i, \pm\dfrac{i\sqrt{2}}{2}$

35. $z = \pm 2, \pm 2i$

36. $x^2 - 6x + 32$