Solving with Absolute Value

Who knew two little lines could cause so much trouble? Ask someone to solve the equation $3x - 2 = 7$ and they'll say "No problem!" Add just two little lines, and ask them to solve the equation $|3x - 2| = 7$ and you're likely to get a blank stare. Unless, of course, they happen to have taken PreCalculus at UConn.

Most students have seen absolute value before and learned of it as something that "makes things positive." That's one good way of viewing it. The absolute value of -5 is 5, or in other words, taking the absolute value of -5 "makes it positive." If something's already positive, then absolute value signs do nothing to it.

$$|-5| = 5 \qquad |5| = 5$$

Now, where things get confusing is if we *don't know* if the thing inside absolute value signs is positive or negative. This situation comes up when we need to solve equations or inequalities that involve absolute value. For example, if I had $|3x - 2| = 7$, well, I don't know if $3x - 2$ is a positive number or a negative number because I don't yet know what x is. So how can I simplify that? The key to solving equations and inequalities that involve absolute value is to break things down into cases. Let's start with a very simple example.

Let's say I want to solve the equation $|x| = 5$. What might x be? Hopefully your brain is telling you it might be 5, or it might be -5. And you're right. But what did you actually do there, without realizing it? You broke this problem down into cases: if x was a positive number and $|x| = 5$, then $x = 5$. If x was a negative number and $|x| = 5$, then $x = -5$.

That's going to be our big trick with absolute value. We need to break it into cases. *However*, the cases will *not* be "If x is negative or if x is positive." Let's see why by looking at another example, slightly more complicated.

Let's say I want to solve the equation $|x - 1| = 5$. What might x be? Now, what I really care about with absolute value is if the thing inside the absolute value signs is positive or negative. That's what makes a difference! In this case, the "thing" inside the absolute value signs is $(x - 1)$. So, while I don't know if $(x - 1)$ is positive or negative, I can take each case one at a time and see what the possibilities might be:

- If $(x-1)$ is positive, then $|x-1| = x-1$. (Since absolute value makes things positive and it's already positive, nothing changes.) So in that case, instead of having $|x - 1| = 5$, I could just have $x - 1 = 5$ and solve from there to get a solution, $x = 6$.
- If $(x - 1)$ is negative, then we have to think a little bit more. If $(x - 1)$ is a negative number, then $-(x-1)$ must be a positive number (a negative negative is a positive). So that means that if $(x-1)$ is negative, then $|x - 1| = -(x - 1)$. So in that case, instead of having $|x - 1| = 5$, I could just have $-(x - 1) = 5$, which simplifies to $x - 1 = -5$, which simplifies to $x = -4$. So there's another solution.

In all, this equation has two solutions, $x = 6$ or $x = 4$.

Try a few similar problems yourself. Note: one of these is a "trick" question – find it and explain why.

(1) $|3x - 2| = 7$

(2) $|7 - x| = 10$

(3) $8 - |x + 1| = 5$

(4) $|3 - 2x| + 4 = 1$

Absolute value works exactly the same if you're solving an inequality. You just have to break it up into cases, where in one case the thing inside the absolute value is positive, and in the other case the thing inside the absolute value is negative. For example, say we want to solve the inequality $|x - 4| > 5$. First of all, we'll consider if $(x - 4)$ is positive. Actually, let's think about it in just a slightly different way: $(x - 4)$ is positive if x is greater than 4 (we can even throw in if $x \geq 4$ – if $x = 4$ then $x - 4 = 0$, and 0 is not positive or negative. Mathematicians have a word for something that is positive or zero: it's *nonnegative*). So:

- If $x \geq 4$, then $|x - 4| = x - 4$, so our inequality becomes $x - 4 > 5$, which simplifies to $x > 9$.
- If $x < 4$, then $|x - 4| = -(x - 4)$, so our inequality becomes $-(x - 4) > 5$, which simplifies to $-x + 4 > 5$, which simplifies to $-x > 1$, which simplifies to $x < -1$.

Therefore, our solution set to this inequality, the set of all values for x that makes the inequality true, is $x > 9$ or $x < -1$. In interval notation, this is $(-\infty, -1) \cup (9, \infty)$.

Want another example? Thought so. Let's say we want to solve the inequality $|3x + 1| - 2 \leq 5$. We could simplify this first, before dealing with the absolute value: add 2 to both sides and we get: $|3x + 1| \leq 7$. Now, what we're concerned about is when $3x + 1$ is nonnegative and when $3x + 1$ is negative. It's nonnegative when $3x + 1 \geq 0$, which is when $3x \geq -1$, which is when $x \geq -\frac{1}{3}$. Otherwise, it must be negative. So we have our two cases:

- If $x \geq -\frac{1}{3}$, then $|3x + 1| \leq 7$ is the same as $3x + 1 \leq 7$, which we can solve:

$$\begin{aligned} 3x + 1 &\leq 7 \\ 3x &\leq 6 \\ x &\leq 2 \end{aligned}$$

- If $x < -\frac{1}{3}$, then $|3x + 1| \leq 7$ is the same as $-(3x + 1) \leq 7$, which we can solve:

$$\begin{aligned} -(3x + 1) &\leq 7 \\ 3x + 1 &\geq -7 \\ 3x &\geq -8 \\ x &\geq -\frac{8}{3} \end{aligned}$$

So our solution set, all values that make this equation true, is $[-\frac{8}{3}, 2]$. Now your turn to try a few.

(5) $|x + 7| \geq 15$

(6) $|x - \frac{1}{2}| < 5$

(7) $|2x - 8| \leq 10$

(8) $|5x+1| > 1$

Now you have the knowledge, believe it or not, to solve even more complicated things like $|x+1|+|x-2| < 5$. Remember, when there's a variable inside an absolute value sign, you can break it up into cases: when the thing inside the absolute value is positive (or nonnegative) and when it's negative. The factor $(x+1)$ is nonnegative when $x \geq -1$, and negative when $x < -1$. The factor $(x-2)$ is nonnegative when $x \geq 2$, and negative when $x < 2$. So, if we want to consider all possible cases, we have:

- $x < -1$. In this case, both things inside the absolute value are negative, so $|x+1| = -(x+1)$ and $|x-2| = -(x-2)$.
- $-1 \leq x < 2$. In this case, one is negative and one is positive, i.e., $|x+1| = x+1$ and $|x-2| = -(x-2)$.
- $x \geq 2$. In this case, both are positive, so $|x+1| = x+1$ and $|x-2| = x-2$.

So to solve this one, we just have to divide up into those cases:

- $x < -1$. Then our inequality becomes $-(x+1) - (x-2) < 5$, which we can solve:

$$\begin{aligned} -(x+1) - (x-2) &< 5 \\ -x - 1 - x + 2 &< 5 \\ -2x + 1 &< 5 \\ -2x &< 4 \\ x &> -2 \end{aligned}$$

  So if $x < -1$, then the inequality is true if $x > -2$. So we have some solutions, $(-2, -1)$.
- $-1 \leq x < 2$. Then the inequality becomes $(x+1) - (x-2) < 5$:

$$\begin{aligned} (x+1) - (x-2) &< 5 \\ x + 1 - x + 2 &< 5 \\ 3 &< 5 \end{aligned}$$

  This is always true, no matter what x is! So, if x is any value between -1 and 2, the inequality is true. So we have more solutions: $[-1, 2)$.
- $x \geq 2$. Then the inequality becomes $(x+1) + (x-2) < 5$:

$$\begin{aligned} (x+1) + (x-2) &< 5 \\ 2x - 1 &< 5 \\ 2x &< 6 \\ x &< 3 \end{aligned}$$

  So if $x \geq 2$, then the inequality is true if $x < 3$. So our remaining solutions are $[2, 3)$.

Put all of these cases together, and all the values of x that make the inequality true are $(-2,-1) \cup [-1, 2) \cup [2, 3)$, which is just the same as the single interval $(-2, 3)$.

Your turn!

(9) $|x+3|-|x-2|<1$

(10) $|x-5|+|x-7|\geq 10$

(11) $|2x-1|+|x|>4$