Rules of Exponents

There are four main rules of exponents:

(1) You do not talk about rules of exponents!
(2) You DO NOT talk about rules of exponents!
(3) Um...

Okay, bad joke. Let's start over. There are four main rules of exponents. If a, b, and c are any numbers, then

(1) $a^0 = 1$
(2) $a^b a^c = a^{b+c}$
(3) $(a^b)^c = a^{bc}$
(4) $a^{-b} = \frac{1}{a^b}$

These rules pretty much make sense. Let's take the first one on faith for now, and then remember what exponents mean: they mean repeated multiplication. So, for example, if we had a^3 that means $a \cdot a \cdot a$. So, for example,

$$a^3 a^2 = (a \cdot a \cdot a)(a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5$$

so that second rule makes sense. Similarly,

$$(a^3)^2 = (a^3) \cdot (a^3) = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$$

so that third rule makes sense. The fourth rule isn't quite as obvious, but think of it this way. Let's say I'm trying to figure out what a^{-3} is. By the second rule, I know that $a^{-3}a^3$ should equal $a^{-3+3} = a^0 = 1$. So $a^{-3} \cdot a^3 = 1$. Divide both sides by a^3 and you get $a^{-3} = \frac{1}{a^3}$, which is what the fourth rule describes.

Simplify the following:

(1) $x^3 \cdot x^7 \cdot (x^3)^5$
(2) $(x^{-3})(x^h)^2(x)$

Another way of thinking of those negative exponents is that they make the term in question "switch" top and bottom. For example, if we had $\frac{1}{a^{-2}}$, we could simplify it:

$$\frac{1}{a^{-2}} = \frac{1}{\frac{1}{a^2}} = 1 \cdot \frac{a^2}{1} = a^2.$$

In other words, the a^{-2} in the denominator became an a^2 in the numerator.

(3) $q^{-4} \cdot \dfrac{4q}{(q^{-2})^3}$
(4) $\dfrac{3}{y^2}\dfrac{y}{y^{-8}}$

Exponents and fractions can get tricky. What about these?

(5) $\left(\dfrac{2y}{x}\right)^2$
(6) $\dfrac{a}{a^4}\left(\dfrac{3}{a^2}\right)^{-3}$

We've only used whole numbers and variables so far – but exponents work exactly the same with any number. For example, you can have $3^{1.2}3^{\frac{1}{3}} = 3^{1.2+\frac{1}{3}} = 3^{\frac{23}{15}}$.

(7) $z^{-\frac{1}{3}}z^{0.5}$
(8) $(x^{\frac{1}{2}})^{\frac{1}{3}}x^{\frac{1}{4}}$
(9) $(x^{\sqrt{2}})^{\sqrt{2}}$

How do we interpret these exponents that aren't whole numbers? Well, we can figure it out. If $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}\cdot 2} = x^1 = x$, that means $x^{\frac{1}{2}}$ is a number where if you square it you get x. In other words, $x^{\frac{1}{2}}$ is the square root of x, $x^{\frac{1}{2}} = \sqrt{x}$. Similarly, $x^{\frac{1}{3}}$ is the third root of x, $\sqrt[3]{x}$ (the third root of x is a number such that if you cube it you get x).

Now, please do not mix up negative numbers, negative exponents, fractions, and fractional exponents. Each of the following is a real mistake that has been seen in precalculus in the past. Notice how they're all incorrect applications of the rules and figure out why.

COMMON MISTAKES

a) $x^{-2} = x^{\frac{1}{2}}$
b) $x^{-2} = \sqrt{x}$
c) $(-x)^2 = \frac{1}{x^2}$
d) $x^{\frac{1}{2}} = (-x)^2$
e) $\frac{1}{x^{\frac{1}{2}}} = x^2$
f) $\frac{1}{x^{-2}} = x^{\frac{1}{2}} = \sqrt{x}$

Try simplifying the following. Avoid those common mistakes.

(7) $\left(\frac{1}{x}\right)^{-\frac{1}{2}}$

(8) $\left(\frac{1}{x^{-2}}\right)^{\frac{1}{2}}$

(9) $\left(\frac{x^{-\frac{1}{2}}}{x^2}\right)^2$

Now notice something. Everything we've done so far involves only multiplication, division, and exponents. That's because addition and subtraction don't mix so well with exponents. They're like cats and dogs. Oil and water. Ice cream and ketchup. Things are never as simple as you'd like them to be. Everyone *wishes* that the following were true, but IT'S NOT:

$$(a+b)^c = a^c + b^c \qquad (\leftarrow \text{NOT TRUE!})$$

Look at an example:

$$(3+5)^2 = 8^2 = 64$$
$$3^2 + 5^2 = 9 + 25 = 34$$

See? They're different. We could expand something like $(x+y)^2$, but it would take more work than that:

$$(x+y)^2 = (x+y)\cdot(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2.$$

If the exponent is bigger, it gets even more complicated. I'll leave out the intermediate steps, but:

$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$$

Basically, what you want to remember is that exponents and addition and subtraction just don't play well together. If you're simplifying exponents, you can't distribute them over addition. But, as always, you can treat something in parentheses like a term all unto itself. So, for example:

$$\frac{x^3(x+2)^{-3}(x-1)^{\frac{1}{2}}}{x+2} = \frac{x^3\sqrt{x-1}}{(x+2)^3(x+2)} = \frac{x^3\sqrt{x-1}}{(x+2)^4}$$

And that's about as simple as it can get. You try a few.

(10) $\dfrac{(x+2)(x-3)}{x(x+2)^{-2}(x-3)^{1/2}}$

(11) $(x+1)^{1-\pi}(x+1)^{1+\pi}$

(12) $\left(\dfrac{x^{4/3}x^3}{x^{-2}}\right)^3\left(\dfrac{x^{-1}}{x}\right)$

(13) $\left(\dfrac{(x-1)^{-2}(x+1)^3}{x^2-1}\right)^2$