Inverse Functions

Inverse functions do the opposite of functions. If a function takes an input and adds 5 to it, its inverse will take an input and subtract 5 from it. If a function divides a number by 17, its inverse will multiply it by 17. If $f(3) = 5$, that means f takes in 3 and spits out 5. It's inverse, then, should take in 5 and spit out 3. In other words, $f^{-1}(5) = 3$. It's like the outputs of f are the inputs of f^{-1}, and vice versa. Mathematically, this is best expressed using the idea of compositions of functions.

The inverse of f is the function f^{-1} such that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

This makes sense according to the intuitive definition first given above. If you take x and plug it into f, you get out $f(x)$. Then take $f(x)$ and plug it into f^{-1}, and you get $f^{-1}(f(x))$, which is the same as $(f^{-1} \circ f)(x)$. On the other hand, f^{-1} should do the opposite of f. Since f turned x into $f(x)$, f^{-1} should turn $f(x)$ back into x. So if you compose the two, you end up back where you started, with x. Putting this all together says that $(f^{-1} \circ f)(x) = x$. A similar argument tells us that $(f \circ f^{-1})(x) = x$ as well. Armed with just this knowledge, you should be able to answer the following questions.

(1) Let $f(x) = x^3 - 5$. Which of the following is true?
- (a) $f^{-1}(2) = 3$
- (b) $f^{-1}(0) = 2$
- (c) $f^{-1}(-4) = 1$
- (d) $f^{-1}(1) = 4$

(2) Let's say $g(2) = 5$. Which of the following might be the inverse of g?
- (a) $g^{-1}(x) = x + 3$
- (b) $g^{-1}(x) = x - 3$
- (c) $g^{-1}(x) = 2x - 7$
- (d) $g^{-1}(x) = x$

(3) Let $f(x) = \frac{x-3}{2}$. Which of the following is the inverse of f?
- (a) $f^{-1}(x) = \frac{x+3}{2}$
- (b) $f^{-1}(x) = \frac{2}{x-3}$
- (c) $f^{-1}(x) = \frac{3-x}{2}$
- (d) $f^{-1}(x) = 2x + 3$

Sometimes we want to find the formula for the inverse of a function. For example, we might know that $f(x) = x^3 - 3$ and try to find a formula for f^{-1}. There's a simple method for this. Since f takes in x and spits out $y = x^3 - 3$, we want something that takes in $y = x^3 - 3$ and spits out x. So what must you do to y in order to turn it into x? Since $y = x^3 - 3$, we know that $y + 3 = x^3$, and so we know that $\sqrt[3]{y+3} = x$. Therefore, the inverse function should take in y and spit out $\sqrt[3]{y+3}$. However, since we usually let x represent the input of a function (whether it's f or f^{-1}), we'll typically change that y into an x and write $f^{-1}(x) = \sqrt[3]{x+3}$.

That kind of procedure works in general:

(1) Write the equation $y = f(x)$.
(2) Solve for x in terms of y.
(3) Interchange all x's and y's. The resulting formula will represent $f^{-1}(x)$.

Here's another example. If $g(x) = \frac{x+1}{2x-5}$ and we want to find g^{-1}, we would do the following.

(1) Set $y = \frac{x+1}{2x-5}$.

(2) Solve for x in terms of y:

$$\begin{aligned} y &= \frac{x+1}{2x-5} \\ y(2x-5) &= x+1 \\ 2xy-5y &= x+1 \\ 2xy-x &= 1+5y \\ x(2y-1) &= 1+5y \\ x &= \frac{1+5y}{2y-1} \end{aligned}$$

(3) Write $g^{-1}(x) = \frac{1+5x}{2x-1}$.

A helpful trick in that problem – if you're trying to solve for some variable, say z, then it might help to get all the terms with a z in them to one side of the equation, factor out a z, and then divide. In this problem, we got all of the terms with an x in them to one side (so we had $2xy - x$ on that side), then factored out an x, and divided by what was left over (which was $2y - 1$).

Now you try out this procedure. Find the inverse function, f^{-1}, for each of the following functions. If you want to double-check your work, you can always compose f and f^{-1} to make sure that $(f \circ f^{-1})(x) = x$.

(1) $f(x) = \frac{x}{2} - 3$

(2) $f(x) = x^3 + 2$

(3) $f(x) = \sqrt[3]{x+1}$

(4) $f(x) = \dfrac{2x-3}{x-1}$

You also read off information about an inverse function from the graph of a function. Let's see what the relationship is. Let's say $f(x) = x^3 + 3$. Then, for example, $f(1) = 4$. Right away we know that $f^{-1}(4) = 1$, since inverse functions "do the opposite." Now, the fact that $f(1) = 4$ means that the point $(1, 4)$ is on the graph of f, and similarly, since $f^{-1}(4) = 1$, we know that the point $(4, 1)$ is on the graph of f^{-1}. Likewise, since $f(2) = 11$, the point $(2, 11)$ is on the graph of f and the point $(11, 2)$ is on the graph of f^{-1}. So to get from the graph of f to the graph of f^{-1} (or vice versa), you simply have to switch the x's and y's. This corresponds to reflecting over the diagonal line $x = y$:

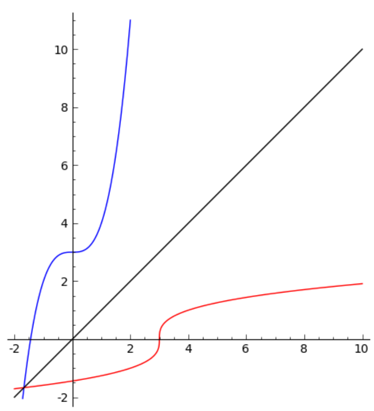

FIGURE 1. The line $y = x$ is in black. The graph of f^{-1} (red) is the reflection of the graph of f (blue) over the line $y = x$.

Now you can answer some questions with your newfound knowledge.

(5) Let f be given by the following graph. What is $f^{-1}(5)$? What is $f^{-1}(1)$?

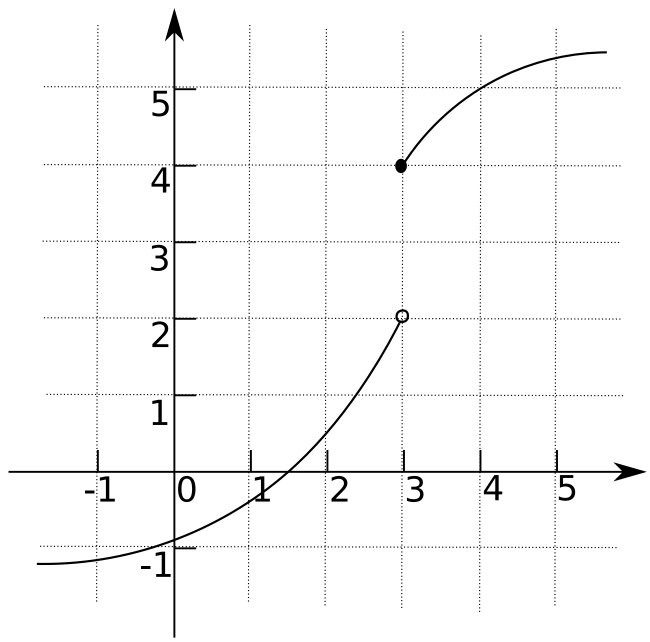

Figure 2.

(6) Let g be given by the following graph. Sketch a graph of g^{-1}.

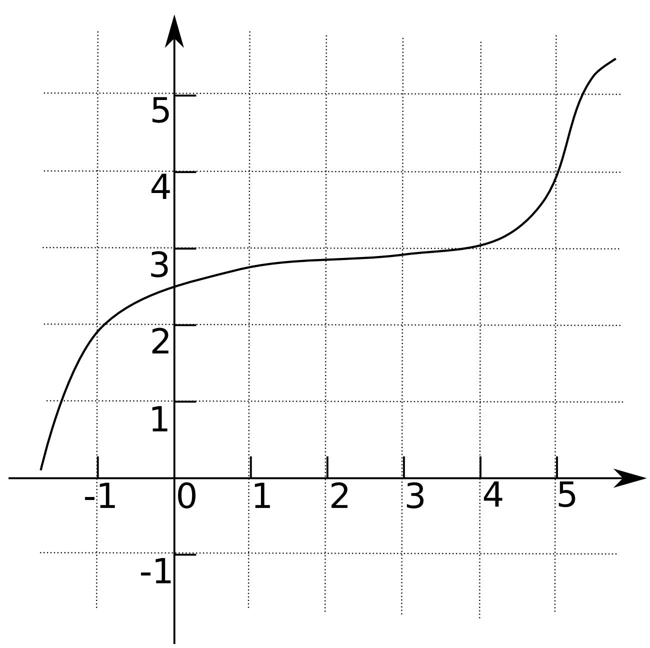

Figure 3.