Rational Functions

Rational functions. That's what you get when you take a polynomial and divide it by another polynomial. Something like $\frac{x^4-3x^7+x-2}{9-3x^7}$. Throw in even just one other thing, like a $\sqrt{x}$ or a e^x or a $\sin x$ and it's no longer called a rational function.

Rational functions may have several distinctive features. They can have zeros (where they cross the x-axis), they can have vertical asymptotes (where it looks like they shoot up to infinity or down to negative infinity in the "y" direction), they can have horizontal asymptotes (where they get closer and closer to some y-value as x gets really big in the positive or negative direction), they can have holes (infinitesimally small spots where they're not defined), and they can have slant asymptotes (we're not going to talk about these much in this class). Let's take a look at these features one at a time.

Zeros. Like any function, a rational function f crosses the x-axis at any point where $f(x) = 0$. If you have a rational function like, say, $g(x) = \frac{x^2+3x-4}{x^3+x^2}$, how do you solve $g(x) = 0$? Well, you set up the equation:

$$\frac{x^2+3x-4}{x^3+x^2} = 0$$

and then you can multiply by the denominator on both sides, which leaves you with:

$$x^2 + 3x - 4 = 0,$$

which we can solve in this case to get $x = -4, 1$. The lesson? To find where a rational function $g(x) = 0$, you just have to pay attention to where the numerator equals 0. There's a little caveat here, but we won't talk about it until we get to talking about holes.

Find the zeros of the following rational functions.

(1) $f(x) = \dfrac{x^3+x^2-2x}{x^2-9}$

(2) $g(x) = \dfrac{5x+2}{x^5+3x+1}$

(3) $h(x) = \dfrac{4}{x^2-x}$

Vertical Asymptotes. You get a vertical asymptote when $f(x)$ approaches $\pm\infty$ as x approaches some particular value. Take the simple rational function $f(x) = \frac{1}{1-x}$. If $x = 1.1$, then $f(x) = 10$. If $x = 1.01$, then $f(x) = 100$. If $x = 1.001$, then $f(x) = 1000$. As the value for x gets closer to 1, the value for $f(x)$ gets bigger and bigger, i.e., approaches infinity. You can draw a picture by plotting those points ((1.1., 10), (1.01, 100), (1.001, 10000)). Connect the dots and you can see that the function has a vertical asymptote at $x = 1$; in other words, the graph of the function gets really really really really close to the line $x = 1$ as you go up the graph at that point. (Note: the "vertical asymptote" of the function is technically the line that the graph of the function approaches, although many people are loose with how that language is used.) Look on the other side of the line $x = 1$. There, we have values of x like $.9, .99, .999$, and so on. Plot the corresponding points and you will find that $f(x)$ approaches negative infinity as x approaches 1 on the left side. We often write that in shorthand, "as $x \to 1^-$, $f(x) \to -\infty$." The minus sign by the 1 indicates that we're talking about the left hand side of $x = 1$. On the other side, we would write "as $x \to 1^+$, $f(x) \to \infty$."

Now, notice that the function f is not actually defined at the spot where it has a vertical asymptote. In fact, a function can only have vertical asymptotes where it's undefined. For a rational function, of course, that's where the denominator might be zero. Take our previous example, $g(x) = \frac{x^2+3x-4}{x^3+x^2}$. The function g might have a vertical asymptote at any place where the denominator equals 0. So we should solve:

$$x^3 + x^2 = 0 \implies x^2(x+1) = 0 \implies x = 0, -1.$$

Notice that we said it *might* have a vertical asymptote at those spots? That's because it might not, as well. Let's look at two examples, one where there is a vertical asymptote and one where there isn't, and find the difference.

Example 1 Let $f(x) = \frac{x^2-2}{x-1}$. This function might have a vertical asymptote at $x = 1$. Does it? Let's play around...if $x = 1.1$, then $f(x) = \frac{1.1^2-2}{.1} = \frac{-.79}{.1} = -7.9$. If $x = 1.01$, then $f(x) = \frac{1.01^2-2}{.01} = \frac{-.9799}{.01} = -97.99$.

As the values of x get closer to 1, the values of $f(x)$ blow up and get bigger and bigger. That's the behavior of an asymptote, so apparently, f has a vertical asymptote at 1.

Example 2 Let $g(x) = \frac{x^2-1}{x-1}$. This function might have a vertical asymptote at $x = 1$. Does it? Let's play around...if $x = 1.1$, then $f(x) = \frac{1.1^2-1}{.1} = \frac{.21}{.1} = 2.1$. If $x = 1.01$, then $f(x) = \frac{1.01^2-1}{.01} = \frac{.0201}{.01} = 2.01$. Now, as the values of x are getting closer to 1, the values of $f(x)$ aren't blowing up, they're staying right around 2. So no vertical asymptote!

The crucial difference here comes if we look at g in simplified form: $g(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$. The $x - 1$ on the bottom "disappeared." The lesson? If the denominator is zero, then you have a vertical asymptote, unless the factor that made the denominator zero disappears completely when you simplify the rational function.

Find any vertical asymptotes of the following rational functions.

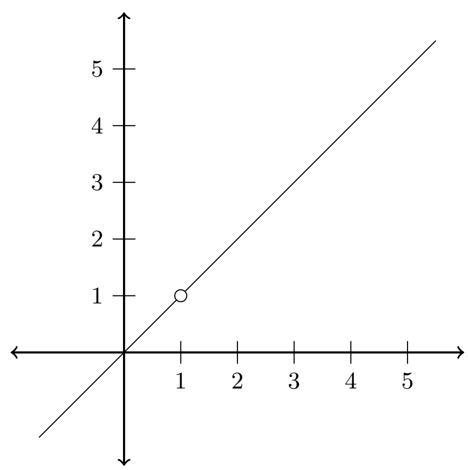
(4) $f(x) = \dfrac{x^3 + x^2 - 2x}{x^2 - 9}$

(5) $h(x) = \dfrac{4}{x^2 - x}$

(6) $q(x) = \dfrac{x^2 - 3x}{x^3 + 2x^2 - 15x}$

(7) $r(x) = \dfrac{x^2 - 4}{x^2 - 4x + 4}$

Holes. So what is it going on when factors cancel? We don't get a vertical asymptote if they cancel all the way, but what do we get? Let's take a very simple example: $f(x) = \frac{x(x-1)}{x-1}$. Clearly, this function can be simplified by canceling the $x - 1$'s to get the function $g(x) = x$. So the function f, where $f(x) = \frac{x(x-1)}{x-1}$, is pretty much the same as the function g, where $g(x) = x$. The only difference is that $f(1) = DNE$ and $g(1) = 1$. Basically, they're the same, except f has a *hole* right at $x = 1$. The way we draw that is like so:



Find any holes of the following rational functions.

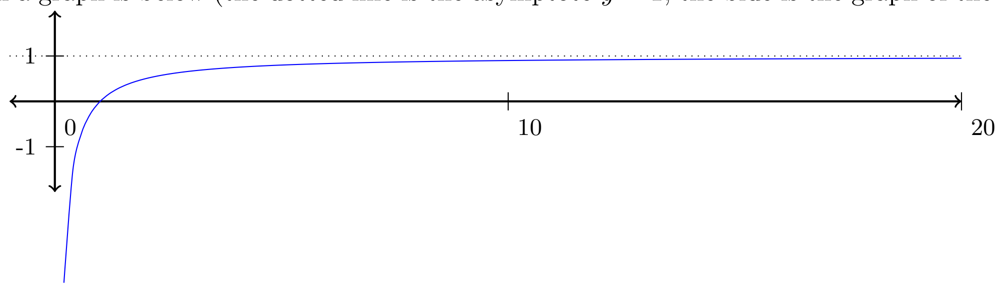
(8) $f(x) = \dfrac{x^3 + x^2 - 2x}{x^2 - 9}$

(9) $h(x) = \dfrac{4}{x^2 - x}$

(10) $q(x) = \dfrac{x^2 - 3x}{x^3 + 2x^2 - 15x}$

(11) $r(x) = \dfrac{x^2 - 4}{x^2 - 4x + 4}$

Horizontal Asymptotes. A horizontal asymptote is a horizontal line that the graph of a function approaches as x tends toward ∞ or $-\infty$. In shorthand, you have a horizontal asymptote if "as $x \to \infty$, $f(x) \to L$" or "as $x \to -\infty$, $f(x) \to L$," where L will be some number. Let's look at a simple example. Take the function $f(x) = \frac{x-1}{x}$. What happens as values of x get bigger and bigger? If $x = 10$, then $f(x) = \frac{9}{10} = .9$. If $x = 100$, then $f(x) = \frac{99}{100} = .99$. If $x = 1000$, then $f(x) = \frac{999}{1000} = .999$. That is, the values of x are getting bigger (10, 100, 1000), and the values of $f(x)$ are getting closer and closer to 1 (.9, .99, .999). What this looks like on a graph is below (the dotted line is the asymptote $y = 1$, the blue is the graph of the function).



There's more to it, but the long and short of it with horizontal asymptotes is that, in polynomials, as x gets larger, the large degree terms matter more than the small degree terms. Think about the polynomial $f(x) = x^4 + 10x + 50$. Sure, when x is a small number, the $10x + 50$ matters. For example, if $x = 1$, then $f(1) = 1 + 10 + 50 = 61$, and if we got rid of the $10x + 50$ part and considered just $g(x) = x^4$, we'd have $g(1) = 1$. But if x is a large number, like 1000, then the $10x + 50$ matters less, since $f(1000) = 1000000000000 + 10000 + 50 = 1000000010050$, and $g(1000) = 1000000000000$. Think of it this way: if you have \$1000000000000 (that's a trillion dollars), then would you really care if you lost ten thousand? The moral of the story is that when you're trying to find horizontal asymptotes, only the largest degree terms matter on top and bottom.

Let's look at an example. Take, say, $h(x) = \dfrac{3x^7 - 12x^2 + 9x}{x^9 - 120x + 3}$. Then, for the purposes of horizontal asymptotes, we can just look at the largest degree terms on top and bottom and figure this function is like $\frac{3x^7}{x^9} = \frac{3}{x^2}$. And what happens to that as x gets bigger and bigger? It gets closer and closer to 0, so there is a horizontal asymptote at 0.

Another example. Let's say $v(x) = \dfrac{3x^7 - 12x^2 + 9x}{18x^7 + 1}$. Then, for the purposes of horizontal asymptotes, this is like $\frac{3x^7}{18x^7} = \frac{3}{18}$. So for this function, there is a horizontal asymptote at $\frac{3}{18}$.

A final example. If $u(x) = \dfrac{3x^7 - 12x^2 + 9x}{103x^4 + 99x}$, then when looking for horizontal asymptotes, we can just consider $\frac{3x^7}{103x^4} = \frac{3x^3}{103}$. As x gets larger, this will get larger and larger, and not get close to any particular number. So this function has no horizontal asymptote.

Last thing: notice that to find vertical asymptotes, it's nice to have the function factored on top and bottom, while to find horizontal asymptotes, it's nice to not have it factored.

Find any horizontal asymptotes of the following rational functions.

(12) $f(x) = \dfrac{2x^3 - x}{5x^3 - 2x^2 + 1}$

(13) $g(x) = \dfrac{(x-1)(x+2)}{x(x+1)(x-3}$

(14) $h(x) = \dfrac{3x^3 - x + 5}{250x^2 + 60x}$

Now put it all together. For the two functions below, find all zeros, vertical asymptotes, holes, and horizontal asymptotes. Then graph the function. It may help to plot several points in addition to finding the above information.

(15) $g(x) = \dfrac{5}{3 - x}$

(16) $f(x) = \dfrac{3x(x-1)}{x^2 + x - 2}$