Solving Inequalities

Solving inequalities is much like solving equations. Do the same thing to both sides and you usually won't run into trouble. There's one big difference:

When solving an inequality, if you multiply or divide both sides by a negative number, then you must switch the direction of the inequality.

This make sense if you use numbers. We know that $3 < 5$, but if we multiply both sides by negative 1, then $-3 < -5$ is false. Switching the direction of the inequality makes it true: $-3 > -5$. Solve the following inequalities:

(1) $x + 5 < 1$

(2) $1 - 2x \geq 0$

(3) $3 + \frac{x}{2} < 6$

(4) $4 - 10x < 2(1 - x)$

(5) $-5x + 1 \leq 2 - x$

(6) $2x + 1 > \frac{7}{2} + \frac{4}{3}x$

(7) $3 < \frac{x+2}{3} \leq 5$

(8) $4 > 2 - x > 3$

It is possible to have an inequality that has no solution, such as $-2 + x > x + 1$, or an inequality that is true for ALL x, such as $5 + 2x \leq 2x + 10$. See why those examples work?

Some more complicated inequalities require a little more in the way of tricks. Take, for example, the inequality

$$x^2 - 5x + 4 > 0.$$

Just like if this were an equation, factoring would be helpful:

$$(x - 4)(x - 1) > 0.$$

Then we must realize that this is really asking us: "when is the product $(x - 4)$ times $(x - 1)$ a positive number?" Remember your positive/negative rules? It must be a positive number if $(x - 4)$ and $(x - 1)$ are both positive or both negative, and it's a negative number if one is positive and the other is negative. The best way to handle this is to break it down into cases. We can do this by figuring that:

- $(x - 1)$ is negative if $x < 1$ and positive if $x > 1$.
- $(x - 4)$ is negative if $x < 4$ and positive if $x > 4$.

So we can cover all of the options if we divide up into cases of $x < 1$, $1 < x < 4$, and $x > 4$.

(1) If $x < 1$, then $(x - 1)$ is negative and $(x - 4)$ is negative, so the product $(x - 4)(x - 1)$ is a positive number, so $(x - 4)(x - 1) > 0$ is true.
(2) If $1 < x < 4$ then $(x - 1)$ is positive and $(x - 4)$ is negative, so the product $(x - 4)(x - 1)$ is a negative number, so $(x - 4)(x - 1) > 0$ is false.
(3) If $x > 4$, then $(x - 1)$ is positive and $(x - 4)$ is positive, so the product $(x - 4)(x - 1)$ is a positive number, so $(x - 4)(x - 1) > 0$ is true.

Therefore, the inequality is true if $x < 1$ or if $x > 4$.

This same kind of analysis can be done with more than two factors, or with rational expressions, like $\frac{(x-2)(x+5)}{x+7}$. Give it a try.

(9) $\dfrac{x^2-9}{x+1} < 0$

(10) $x^2 + 5x + 1 < -5$

(11) $\dfrac{(x-2)(x+4)}{x+7} > 0$

(12) $\dfrac{(x-2)(x+4)}{x+7} \geq 0$