Solving Trig Equations, Part I

The first types of trig equations you learn to solve are mostly an exercise in using the unit circle. You start with equations like $\sin x = \frac{1}{2}$. If you have an equation in this nice form, "trig function = a number," you can go to the unit circle to find your solutions. In this example, remember that the sine of an angle is represented by the y-coordinate of that angle on the unit circle, so we find where the y-coordinate equals $\frac{1}{2}$. (Needless to say, this requires a good grasp and knowledge of the unit circle, so if you're still uncomfortable with it, get comfortable with it fast!) Those angles with a y-coordinate of $\frac{1}{2}$ are at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, if we restrict ourselves to only the interval $[0, 2\pi]$. However, other possible solutions to this equation include $-\frac{7\pi}{6}$, or $\frac{13\pi}{6}$, or $\frac{41\pi}{6}$... of course there are an infinite number of solutions to this equation. For that reason, we usually restrict our attention to solutions in the interval $[0, 2\pi]$, but sometimes we may want to describe *all* of the possible solutions, for which this kind of notation is helpful:

$$x = \frac{\pi}{6} + 2k\pi \text{ or } \frac{5\pi}{6} + 2k\pi \text{ for any integer } k.$$

This way, we can describe all possible solutions in a succinct manner.

Since sine, cosine and tangent are easiest to read off the unit circle, if you're dealing with other trig functions it's often easiest to rewrite in terms of these three. For example, if we were to solve $\sec t = -2$, we might want to rewrite this as $\frac{1}{\cos t} = -2$, and then apply algebra to find that $\cos t = -\frac{1}{2}$, so $t = \frac{\pi}{3} + 2k\pi$ or $t = \frac{4\pi}{3} + 2k\pi$ for any integer k.

Here are some simple equations to get started with. Find *all* solutions.

(1) $\cos t = -\frac{\sqrt{3}}{2}$

(2) $\csc t = 2$

(3) $\cot t = -1$

(4) $\sin t = -1$

(5) $\tan t = \sqrt{3}$

Sometimes an equation doesn't look as simple as "trig function = a number." In that case, your job is to reduce it to that form first using plain algebra. Here's an example – this time we'll restrict our solutions to the interval $[0, 2\pi]$.

$$\begin{aligned} \frac{\tan t - 3}{2 - 4\tan t} &= 1 \\ \tan t - 3 &= 2 - 4\tan t \\ 5\tan t &= 5 \\ \tan t &= 1 \\ t &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

Find all solutions in the interval $[0, 2\pi]$, or explain why no solutions exist.

(6) $\sqrt{2}\cos t = -1$

(7) $\dfrac{3 + 2\sin t}{5} = \sin t$

(8) $1 = \dfrac{1 + 3\cos t}{5\cos t - 2}$

(9) $\dfrac{6\sec t + 2}{2\sec t - 1} = 2$