Practice Problems - Exam 1 (Due Mon, May 19)

Math 1060Q – Summer 2014

Professor Hohn

1. Suppose f and g are the functions completely defined by the tables below:

x	$f(x)$
1	-2
-3	1
5	-4

x	$g(x)$
-4	1
-2	-3
1	5

Make a table of $f \circ g$ and a table of $g \circ f$.

Solution:

x	$(f \circ g)(x)$
-4	$f(g(-4)) = f(1) = -2$
-2	$f(g(-2)) = f(-3) = 1$
1	$f(g(1)) = f(5) = -4$

x	$(g \circ f)(x)$
1	$g(f(1)) = g(-2) = -3$
-3	$g(f(-3)) = g(1) = 5$
5	$g(f(5)) = g(-4) = 1$

2. Find the maximum value of $5 - 8x - 2x^2$.

Solution: Notice that the coefficient in front of the x^2 term is negative, and thus, our parabola will have a maximum value. To find this maximum value, we must first find the vertex of the parabola. A quick way to find the vertex is to rewrite our quadratic function in standard form $a(x-h)^2 + k$.

$$\begin{aligned} 5 - 8x - 2x^2 &= -2x^2 - 8x + 5 \\ &= -2(x^2 + 4x + \qquad) + 5 \\ &= -2(x^2 + 4x + 4 - 4) + 5 \\ &= -2(x^2 + 4x + 4) - 4(-2) + 5 \\ &= -2(x+2)^2 + 8 + 5 \\ &= -2(x+2)^2 + 13 \end{aligned}$$

The vertex is then $(-2, 13)$ and the maximum value that this quadratic function will attain is 13.

3. Let $f(x) = \dfrac{7x+8}{x+4}$.

   (a) Find the domain of f.

   Solution: The x-values (or input values) that make sense for this function are all real numbers except for $x = -4$ (since we don't want to divide by zero). Thus, the domain of our function f is $(-\infty, -4) \cup (-4, \infty)$.

   (b) Find the range of f.

   Solution: Recall that the range of f is the same as the domain of f^{-1}. Thus, we only need to find the domain of f^{-1} (which is solved in the next part). From our f^{-1} function, the domain is all real numbers except $y = 7$. In interval notation, our range is $(-\infty, 7) \cup (7, \infty)$.

   (c) Find a formula for f^{-1}.

   Solution: First, we set $y = \dfrac{7x+8}{x+4}$, and then we solve for x to find the inverse function $f^{-1}(y)$.

$$\begin{aligned} y &= \frac{7x+8}{x+4} \\ y(x+4) &= 7x+8 \\ xy+4y &= 7x+8 \\ xy-7x &= 8-4y \\ x(y-7) &= 8-4y \\ x &= \frac{8-4y}{y-7} \end{aligned}$$

   Thus, $f^{-1}(y) = \dfrac{8-4y}{y-7}$.

   (d) Find the domain of f^{-1}.

   Solution: The domain of the function $f^{-1}(y)$ is all real numbers except $y = 7$. In interval notation, our domain is $(-\infty, 7) \cup (7, \infty)$.

   (e) Find the range of f^{-1}.

   Solution: Recall that the range of f^{-1} is the domain of f. Hence, the range of f^{-1} is $(-\infty, -4) \cup (-4, \infty)$.

4. Write $\dfrac{27^{100}}{9^{45}}$ as a power of 3.

   Solution: We want to write the following as a power of 3 so we must first make 27 into a power of 3 and 9 into a power of 3. Then, we will follow the rules of exponents to get our final answer.

$$\begin{aligned}
\frac{27^{100}}{9^{45}} &= \frac{(3^3)^{100}}{(3^2)^{45}} \\
&= \frac{3^{3\cdot 100}}{3^{2\cdot 45}} \\
&= \frac{3^{300}}{3^{90}} \\
&= 3^{300} \cdot 3^{-90} \\
&= 3^{300-90} \\
&= 3^{210}
\end{aligned}$$

Therefore, $\frac{27^{100}}{9^{45}} = 3^{210}$.

5. Give an example of a function that is neither even nor odd, and explain why it is neither.

Solution: Recall that a function f is odd if $f(x) = -f(-x)$ and a function f is even if $f(x) = f(-x)$. So, we want to find a function f that is neither which means that we want to find a function f such that $f(x) \neq -f(-x)$ (not odd) and $f(x) \neq f(-x)$ (not even). Let $f(x) = x^2 - 5x + 1$. To show that f is not odd, we will show that $f(x) \neq -f(-x)$.

$$-f(-x) = -((-x)^2 - 5(-x) + 1) = -(x^2 + 5x + 1) = -x^2 - 5x - 1 \neq f(x).$$

Since $-f(-x) \neq f(x)$, f is not odd.

Similarly,

$$f(-x) = (-x)^2 - 5(-x) + 1 = x^2 + 5x + 1 \neq f(x).$$

Since $f(-x) \neq f(x)$, f is not even. Hence, f is a function that is neither even nor odd.

6. Find a number t such that the line containing the points $(t, -5)$ and $(-3, 5)$ is perpendicular to the line that contains the points $(-5, 7)$ and $(1, 11)$.

Solution: First, we need to find the slope that is perpendicular to the line containing the points $(-5, 7)$ and $(1, 11)$. Then, we can create a line through the point $(-3, 5)$, and finally find a number t on that line.

Slope through the points $(-5, 7)$ and $(1, 11)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 7}{1 - (-5)} = \frac{4}{6} = \frac{2}{3}.$$

The slope we want for our equation through the point $(-3, 5)$ will be

$$\hat{m} = -\frac{1}{m} = -\frac{3}{2}.$$

Since we have a slope and a point, we can use point-slope form for our line to get

$$y - 5 = -\frac{3}{2}(x - (-3)).$$

So, we now have a line that is perpendicular to the points $(-5, 7)$ and $(1, 11)$ and through the point $(-3, 5)$. To find t, we need to plug in $y = -5$ into our equation $y - 5 = -\frac{3}{2}(x + 3)$.

$$\begin{aligned} -5 - 5 &= -\frac{3}{2}(t + 3) \\ -10 &= -\frac{3t}{2} - \frac{9}{2} \\ -10 + \frac{9}{2} &= -\frac{3t}{2} \\ -\frac{11}{2} &= -\frac{3t}{2} \\ \frac{11}{3} &= t \end{aligned}$$

7. Simplify the expression $\left(\frac{(t^3w^5)^{-3}}{(t^{-3}w^2)^4}\right)^{-2}$.

Solution: We will simplify the expression using the rules of exponents.

$$\begin{aligned} \left(\frac{(t^3w^5)^{-3}}{(t^{-3}w^2)^4}\right)^{-2} &= \left(\frac{t^{-9}w^{-15}}{t^{-12}w^8}\right)^{-2} \\ &= \frac{t^{18}w^{30}}{t^{24}w^{-16}} \\ &= t^{18}w^{30}t^{-24}w^{16} \\ &= t^{-6}w^{46} \\ &= \frac{w^{46}}{t^6} \end{aligned}$$

8. Suppose $g(x) = 3 + \frac{x}{5x - 2}$. Find the formula for g^{-1}.

Solution: First, we set $y = 3 + \frac{x}{5x - 2}$, and then we solve for x to find the inverse function

$g^{-1}(y)$.

$$\begin{aligned} y &= 3 + \frac{x}{5x-2} \\ y(5x-2) &= 3(5x-2) + x \\ 5xy - 2y &= 15x - 6 + x \\ 5xy - 2y &= 16x - 6 \\ 5xy - 16x &= 2y - 6 \\ x(5y-16) &= 2y - 6 \\ x &= \frac{2y-6}{5y-16} \end{aligned}$$

Thus, $g^{-1}(y) = \dfrac{2y-6}{5y-16}$.

9. What is the minimum value of the function f defined by $f(x) = 4x^2 - 8x + 11$? The graph of f is a parabola. Find the vertex of the parabola.

Solution: The minimum (or maximum value) of a quadratic function occurs at the vertex of the parabola. Hence, we need to find the vertex of $f(x) = 4x^2 - 8x + 11$. The vertex is easiest to find when f is in standard form ($f(x) = a(x-h)^2 + k$). We will now rewrite our equation to be in that form.

$$\begin{aligned} f(x) &= 4x^2 - 8x + 11 \\ &= 4(x^2 - 2x \qquad) + 11 \\ &= 4(x^2 - 2x + 1 - 1) + 11 \\ &= 4(x^2 - 2x + 1) - 1(4) + 11 \\ &= 4(x-1)^2 - 4 + 11 \\ &= 4(x-1)^2 + 7 \end{aligned}$$

Hence, the vertex is $(1, 7)$ and the minimum value of this parabola is 7.

10. Let $f(-1) = 10$, $f(2) = 4$, and $f(3) = 2$. Make a table for $g(x)$ where $g(x) = 5f(3x+2) - 1$. Find the domain and range of $g(x)$.

Solution: First, let's draw a table for f.

x	$f(x)$
-1	10
2	4
3	2

We need to find the values that we can input into g for our x column. Since f only allows the inputs $-1, 2, 3$, g must have inputs such that $3x + 2 = -1$, $3x + 2 = 2$ and $3x + 2 = 3$. So, the table for $g(x)$ would be:

x	$g(x) = 5f(3x+2) - 1$
-1	$g(-1) = 5f(3(-1)+2) - 1 = 5f(-1) - 1 = 5 \cdot 10 - 1 = 49$
0	$g(0) = 5f(3(0)+2) - 1 = 5f(2) - 1 = 5 \cdot 4 - 1 = 19$
$\frac{1}{3}$	$g(\frac{1}{3}) = 5f(3(\frac{1}{3})+2) - 1 = 5f(3) - 1 = 5 \cdot 2 - 1 = 9$

The domain of g is $\{-1, 0, \frac{1}{3}\}$ and the range of g is $\{49, 19, 9\}$.

11. Show that for every real number t, the point $(5-3t, 7-4t)$ is on the line containing the points $(2,3)$ and $(5,7)$.

Solution: First, we must find the line that passes through the points $(2,3)$ and $(5,7)$. Then, we will show that the point $(5-3t, 7-4t)$ is on that line.

The slope we are looking for is

$$m = \frac{y_2 - y_2}{x_2 - x_1} = \frac{7-3}{5-2} = \frac{4}{3}$$

Using point-slope form, we have the equation

$$y - 3 = \frac{4}{3}(x-2).$$

Now, we need to show that the point $(5-3t, 7-4t)$ lies on the line $y - 3 = \dfrac{4}{3}(x-2)$. Our point lies on the line if we plug $x = 5-3t$ into our equation and we get $y = 7-4t$.

$$\begin{aligned} y - 3 &= \frac{4}{3}((5-3t) - 2) \\ y - 3 &= \frac{4}{3}(-3t+3) \\ y - 3 &= -\frac{12t}{3} + \frac{12}{3} \\ y - 3 &= -4t + 4 \\ y &= -4t + 7 \end{aligned}$$

Thus, the point $(5-3t, 7-4t)$ is on the line containing the points $(2,3)$ and $(5,7)$.

12. Simplify $\left(\dfrac{xy^{-3}}{x^5y^{-10}z^3}\right)^{-3}$.

Solution:

$$\begin{aligned}\left(\frac{xy^{-3}}{x^5y^{-10}z^3}\right)^{-3} &= \frac{x^{-3}y^9}{x^{-15}y^{30}z^{-9}} \\ &= x^{-3}y^9x^{15}y^{-30}z^9 \\ &= x^{12}y^{-21}z^9 \\ &= \frac{x^{12}z^9}{y^{21}}\end{aligned}$$

13. Find all real numbers x such that $2x^4 - 20x^2 - 22 = 0$.

Solution: The equation $2x^4 - 20x^2 - 22 = 0$ looks similar to a quadratic equation; we will use this idea to help us solve. First, we will put $2x^4 - 20x^2 - 22 = 0$ into a quadratic form. Let $y = x^2$. Then, $x^4 = (x^2)^2 = y^2$. The equation $2x^4 - 20x^2 - 22 = 0$ is equivalent to $2y^2 - 20y - 22 = 0$ where $y = x^2$. We want to find all x that satisfy the equation $2x^4 - 20x^2 - 22 = 0$, so we will first have to find all y that satisfy the equation $2y^2 - 20y - 22 = 0$. We will use factoring to solve for y.

$$\begin{aligned}2y^2 - 20y - 22 &= 0 \\ 2(y^2 - 10y - 11) &= 0 \\ 2(y - 11)(y + 1) &= 0\end{aligned}$$

Thus, either $y - 11 = 0$ or $y + 1 = 0$. If $y - 11 = 0$, then $y = 11$. Similarly, if $y + 1 = 0$ then $y = -1$. Recall that our goal was to find all x that satisfy our equation. Thus, we need to substitute $x^2 = y$ back into our solutions.

$$y = 11 \implies x^2 = 11 \implies x = \pm\sqrt{11}$$

So, two solutions to $2x^4 - 20x^2 - 22 = 0$ are $x = \sqrt{11}$ and $x = -\sqrt{11}$. Now, we need to apply the same idea to $y = -1$.

$$y = -1 \implies x^2 = -1 \implies x = \pm\sqrt{-1}$$

But, we will not have a real number as a solution. Thus, it is not a solution to $2x^4 - 20x^2 - 22 = 0$. Therefore, the solutions to $2x^4 - 20x^2 - 22 = 0$ are $x = \sqrt{11}$ and $x = -\sqrt{11}$.

14. Find two positive numbers whose difference equals 4 and whose product equals 15.

Solution: Let x and y be the two unknown positive numbers. We know from above that $x - y = 4$ and $xy = 15$. Now, we must find x and y. Since $x - y = 4$, $x = y + 4$. We substitute that into the equation $xy = 15$ to get the equation $(y + 4)y = 15$. Let's solve for y.

$$\begin{aligned}y(y + 4) &= 15 \\ y^2 + 4y &= 15 \\ y^2 + 4y - 15 &= 0\end{aligned}$$

We will use the quadratic formula to solve for y.

$$y = \frac{-4 \pm \sqrt{4^2 - 4(1)(-15)}}{2(1)} \implies y = \frac{-4 \pm \sqrt{16+60}}{2} \implies y = \frac{-4 \pm \sqrt{76}}{2}$$

So, we have $y = -2 + \frac{\sqrt{76}}{2}$ (notice that we are only using the positive solution because we were told y was positive). From our equations above, we know that $x = y + 4$. Hence,

$$x = y + 4 \implies x = -2 + \frac{\sqrt{76}}{2} + 4 \implies x = 2 + \frac{\sqrt{76}}{2}.$$

The two numbers are then $2 + \frac{\sqrt{76}}{2}$ and $-2 + \frac{\sqrt{76}}{2}$. Let's verify our solution is correct.

$$x - y = \left(2 + \frac{\sqrt{76}}{2}\right) - \left(-2 + \frac{\sqrt{76}}{2}\right) \implies x + y = 2 + \frac{\sqrt{76}}{2} + 2 - \frac{\sqrt{76}}{2} \implies x - y = 4$$

And, $x - y = 4$. Now, we will check that $xy = 15$.

$$xy = \left(2 + \frac{\sqrt{76}}{2}\right)\left(-2 + \frac{\sqrt{76}}{2}\right) \implies xy = -4 + \left(\frac{\sqrt{76}}{2}\right)^2 \implies xy = -4 + \frac{\sqrt{76}}{4}$$
$$\implies xy = -4 + 19 \implies xy = 15$$

15. Suppose f is a function with domain $[1, 3]$ and range $[2, 5]$. Define functions g and h by

$$g(x) = 4f(x) \quad \text{and} \quad h(x) = f(3x).$$

(a) What is the domain of g?

Solution: The equation $g(x)$ is vertically stretching $f(x)$ by a factor of 4. This action does not effect the domain. Hence, the domain of g will be the same as f: $D_g = [1, 3]$

(b) What is the range of g?

Solution: The equation $g(x)$ is vertically stretching $f(x)$ by a factor of 4. This action effects the range by a scale of 4. Hence, the range of g will be the range of f multiplied by 4: $R_g = [8, 20]$.

(c) What is the domain of h?

Solution: The equation $h(x)$ is horizontally stretching $f(x)$ by a factor of $\frac{1}{3}$. This action effects the domain by a scale of $\frac{1}{3}$. Hence, the domain of h will be the domain of f multiplied by $\frac{1}{3}$: $D_h = [\frac{1}{3}, 1]$.

(d) What is the range of h?

Solution: The equation $h(x)$ is horizontally stretching $f(x)$ by a factor of $\frac{1}{3}$. This action does not effect the range. Hence, the range of h will be the same as f: $R_h = [2, 5]$

16. Fill in the blank.

(a) Let $f(x)$ be a function and x be in the domain of f. Then $f^{-1}(f(x)) = x$.

(b) The equation of the graph $g(x)$ that is obtained by horizontally stretching the graph of $f(x)$ 5 units and by shifting down 7 units is $g(x) = f\left(\frac{1}{5}x\right) - 7$.

(c) The degree of the polynomial $p(x) = 4 + 6x^5 + 3x^2$ is 5.

(d) The function $g(x) = 3x^3 + x$ is a function that is odd (even, odd, or neither).

(e) An example of a polynomial of degree four whose only zeros are -3, 4, and 1 is $p(x) = (x+3)(x-4)(x-1)^2$.