Practice Problems - Exam 2 (Due Tue, May 27)

Math 1060Q – Summer 2014
Professor Hohn

1. True or false.
   (a) $\dfrac{\ln 8}{\ln 2} = 4$

   (a) **F**

   (b) $\cos\left(\dfrac{\pi}{3}\right) = \cos\left(\dfrac{5\pi}{3}\right)$

   (b) **T**

   (c) $\sin(x+y) = \sin(x) + \sin(y)$

   (c) **F**

   (d) $(\log_9 3)\left(\log_5 \dfrac{1}{25}\right) = -1$

   (d) **T**

   (e) $f(\theta) = \cos\theta$ is an even function.

   (e) **T**

2. Show that $2 - \log x = \log\left(\dfrac{100}{x}\right)$ for every positive x.

Solution: Using logarithm rules:

$$\begin{aligned} \log\left(\frac{100}{x}\right) &= \log(100) - \log(x) && \text{(quotient rule)} \\ &= 2 - \log(x) && \text{(since } 10^2 = 100\text{)} \end{aligned}$$

Thus, $\log\left(\dfrac{100}{x}\right) = 2 - \log x$.

3. Let $f(x) = \dfrac{3x^2 + 4x + 1}{2x^2 - 4x + 2}$. Find the vertical asymptotes, end behavior, holes, and zeros of $f(x)$. Sketch $f(x)$.

Solution: First, we will factor our rational function so we can find vertical asymptotes, holes, and zeros.

$$f(x) = \frac{3x^2 + 4x + 1}{2x^2 - 4x + 2} = \frac{(3x+1)(x+1)}{2(x^2 - 2x + 1)} = \frac{(3x+1)(x+1)}{2(x-1)^2}$$

V.A.: To find vertical asymptotes, we are looking for the zeros of the polynomial in the denominator of $f(x)$.

$$2(x-1)^2 = 0 \implies (x-1)^2 = 0 \implies x = 1$$

We have a vertical asymptote at $x = 1$.

E.B.: To find end behavior, we look at the leading term of the polynomial in the numerator and the leading term of the polynomial in the denominator.

$$f(x) = \frac{3x^2 + 4x + 1}{2x^2 - 4x + 2} = \frac{3x^2(1 + \frac{4}{3x} + \frac{1}{3x^2})}{2x^2(1 - \frac{4}{2x} + \frac{2}{2x^2})}$$

As $x \to \infty$, $f(x) \approx \dfrac{3x^2}{2x^2} = \dfrac{3}{2}$. Thus, as $x \to \infty$, $f(x) \to \dfrac{3}{2}$. Similarly, as $x \to -\infty$, $f(x) \approx \dfrac{3x^2}{2x^2} = \dfrac{3}{2}$. Thus, as $x \to -\infty$, $f(x) \to \dfrac{3}{2}$.

Holes: We do not have any holes.

Zeros: To find the zeros of f, we look at the zeros of the polynomial in the numerator of f.

$$(3x+1)(x+1) = 0 \implies 3x + 1 = 0 \quad \text{or} \quad x + 1 = 0 \implies x = -\frac{1}{3}, -1$$

Thus, f intersects the x-axis at the points $(-\frac{1}{3}, 0)$ and $(-1, 0)$.

Before sketching f, it is usually helpful to find the $y-$intercept.

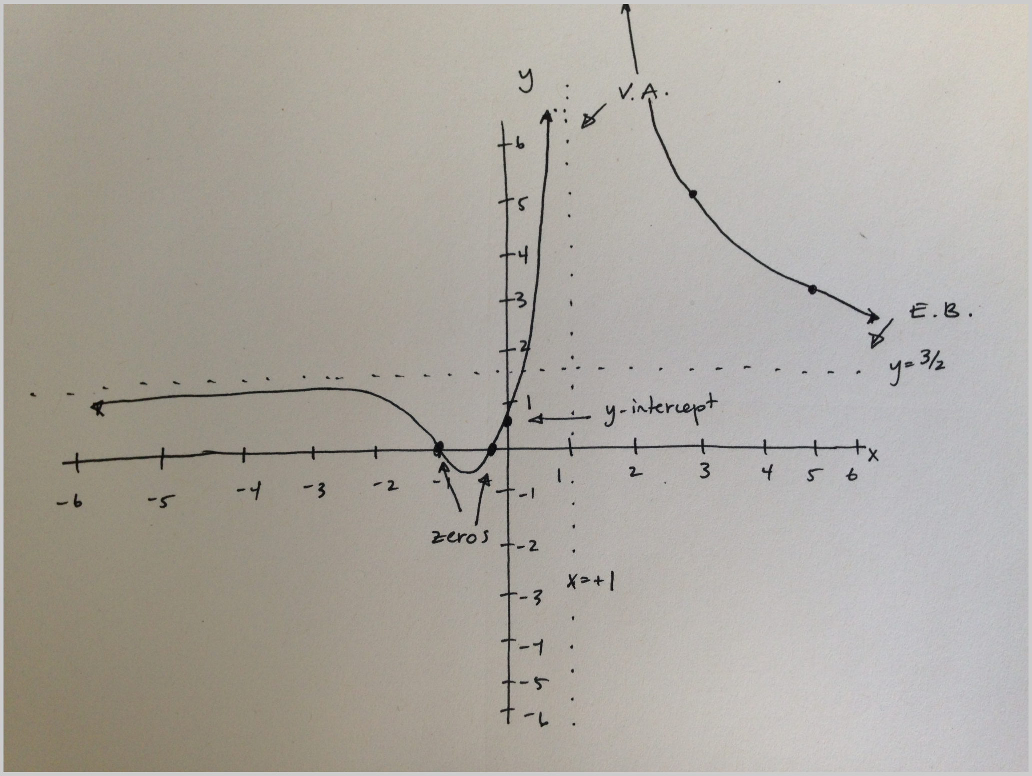
$$f(0) = \frac{3 \cdot 0^2 + 4 \cdot 0 + 1}{2 \cdot 0^2 - 4 \cdot 0 + 2} = \frac{1}{2}$$

In addition, we will check a few points.

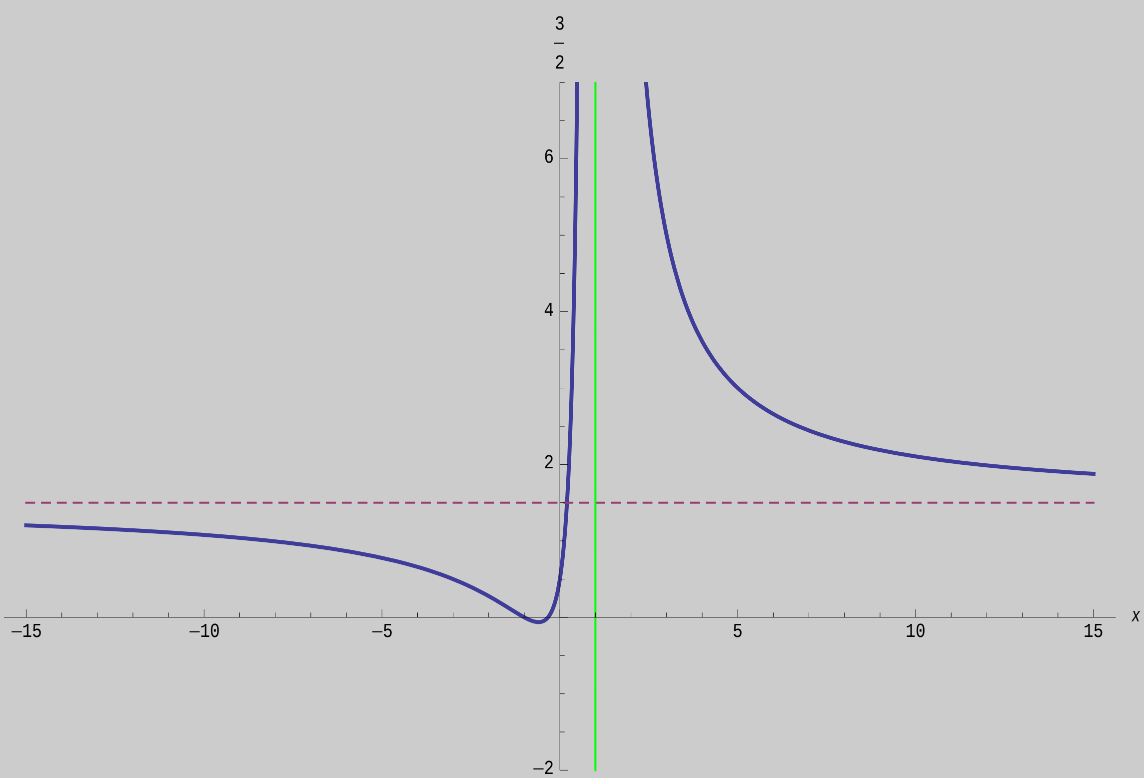
$$f(3) = \frac{27 + 12 + 1}{18 - 12 + 2} = \frac{40}{8} = 5$$

$$f(5) = \frac{3 \cdot 25 + 20 + 1}{2 \cdot 25 - 20 + 2} = \frac{96}{32} = 3$$

Sketch:



Graph:



4. Find the smallest possible positive number x such that $16\sin^4 x - 16\sin^2 x + 3 = 0$.

Solution: Let $\sin^2 x = y$. Then,

$$16\sin^4 x - 16\sin^2 x + 3 = 16y^2 - 16y + 3.$$

Now, we will factor our polynomial.

$$16y^2 - 16y + 3 = (4y-3)(4y-1) = 0$$

We have

$$4y - 3 = 0 \implies 4y = 3 \implies y = \frac{3}{4}.$$

Substituting $y = \sin^2 x$ into our solution for y, we have

$$\sin^2 x = \frac{3}{4} \implies \sin x = \pm\frac{\sqrt{3}}{2}.$$

And, we see that $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. Similarly,

$$4y - 1 = 0 \implies 4y = 1 \implies y = \frac{1}{4}.$$

Substituting $y = \sin^2 x$ into our solution for y, we have

$$\sin^2 x = \frac{1}{4} \implies \sin x = \pm\frac{1}{2}.$$

And, we see that $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Thus, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

5. Find all numbers x such that $\frac{\ln(11x)}{\ln(4x)} = 2$.

Solution:

$$\begin{aligned}
\frac{\ln(11x)}{\ln(4x)} &= 2 \\
\ln(11x) &= 2(\ln(4x)) && \text{(multiplying both sides by } \ln(4x)) \\
\ln(11x) &= \ln((4x)^2) && \text{(power rule)} \\
\ln(11x) &= \ln(16x^2) \\
11x &= 16x^2 \\
16x^2 - 11x &= 0 \\
x(16x - 11) &= 0
\end{aligned}$$

From our equation, we find that $x = 0$ or $16x - 11 = 0$. So, $x = 0, \frac{11}{16}$. Now, since we worked with logarithms, we need to check our answers to make sure that they make sense in our equation. Notice that if we let $x = 0$, $\ln(11 \cdot 0) = \ln(0)$ which is undefined. That is, $x \neq 0$. Thus, our solutions is $x = \frac{11}{16}$.

6. Suppose a colony of 100 cells of the bacteria Precalcitis quadruples in size every two hours.

   (a) Find a function that models the population growth of the colony of bacteria.

   Solution: Since our function is an exponential function, it is of the form $f(x) = cb^x$. We are starting with 100 cells which means $f(0) = 100$. Because the bacteria quadruples in size every two hours, we know that $f(2) = 400$. Our goal is to use this information to find c and b in our equation $f(x) = cb^x$.

   $$f(0) = 100 \implies cb^0 = 100 \implies c \cdot 1 = 100 \implies c = 100$$
   $$f(2) = 400 \implies cb^2 = 400 \implies 100b^2 = 400 \implies b^2 = 4 \implies b = 2$$

   Hence, $f(x) = 100 \cdot 2^x$.

   (b) Approximately how many cells will be in the colony after five hours.

   Solution: Using our equation above, $f(5) = 100 \cdot 2^5 = 100 \cdot 32 = 3200$. Thus, after five hours, the colony will have 3200 cells.

7. Find all numbers x that satisfy $\log_3(x+5) + \log_3(x-1) = 2$.

Solution:

$$\begin{aligned} \log_3(x+5) + \log_3(x-1) &= 2 \\ \log_3((x+5)(x-1)) &= 2 && \text{(product rule)} \\ (x+5)(x-1) &= 3^2 \\ x^2 + 4x - 5 &= 9 \\ x^2 + 4x - 14 &= 0 \end{aligned}$$

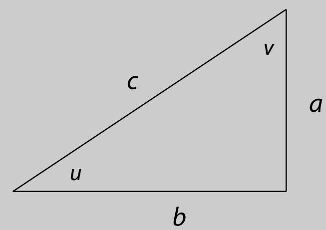
Using the quadratic formula, we see that

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-14)}}{2} \implies x = \frac{-4 \pm \sqrt{72}}{2} \implies x = -2 \pm \sqrt{18} \implies x = -2 \pm 3\sqrt{2}.$$

We are working with logarithms, so we need to check our answer to make sure that it makes sense. Notice that if we let $x = -2 - 3\sqrt{2}$ that $\log_3(x-1)$ is undefined. Thus, our only solution is $x = -2 + 3\sqrt{2}$.

8. Suppose a 19-foot ladder is leaning against a wall, making a 60° angle with the ground. How high up the wall is the end of the ladder?

Solution: Suppose we have the following triangle:



If the ladder is 19 feet long and is leaning against a wall at a 60° angle, then $c = 19$ and $u = 60^\circ$. We are looking for the length of side a.

We know that

$$\sin u = \frac{a}{c} \implies \sin(60^\circ) = \frac{a}{19} \implies 19\sin(60^\circ) = a \implies a = \frac{19\sqrt{3}}{2}$$

9. Suppose y is a number such that $\tan y = -\dfrac{2}{9}$. Evaluate $\tan(-y)$.

Solution: Since $f(y) = \tan(y)$ is an odd function, $f(-y) = -f(y)$. So,

$$f(-y) = \tan(-y) = -\tan(y) = -\left(-\frac{2}{9}\right).$$

Thus, $\tan(-y) = \dfrac{2}{9}$.

10. Create a table showing the endpoints of the radius of the unit circle corresponding to the angles $\dfrac{3\pi}{2}, \dfrac{5\pi}{3}, \dfrac{7\pi}{4}$, and $\dfrac{11\pi}{6}$.

Solution:

Angle	Point on unit circle
$\dfrac{3\pi}{2}$	$\Big(0, -1\Big)$
$\dfrac{5\pi}{3}$	$\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$
$\dfrac{7\pi}{4}$	$\left(\dfrac{\sqrt{2}}{2}, \dfrac{-\sqrt{2}}{2}\right)$
$\dfrac{11\pi}{6}$	$\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$

11. Show that

$$\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta}$$

for all θ except odd multiples of $\frac{\pi}{2}$.

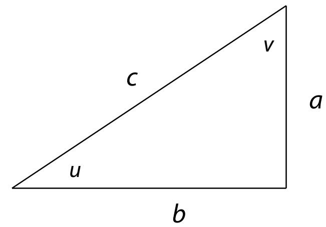
Solution: We will start with the right hand side and show that it is equal to the left hand side.

$$\begin{aligned}
\frac{\tan^2\theta}{1+\tan^2\theta} &= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} \\
&= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} \\
&= \frac{\sin^2\theta}{\cos^2\theta} \div \frac{1}{\cos^2\theta} \\
&= \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\cos^2\theta}{1} \\
&= \sin^2\theta
\end{aligned}$$

Thus,

$$\frac{\tan^2\theta}{1+\tan^2\theta} = \sin^2\theta.$$

12. Use the figure to the right to solve the following:



Suppose $a = 5$ and $b = 8$. Evaluate

(a) $\sin u$

Solution: First, we need to find c. From the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$. Thus, $c^2 = 5^2 + 8^2$ and $c = \sqrt{89}$.

$$\sin u = \frac{\text{opposite}}{\text{hypotenuse}} \implies \sin u = \frac{a}{c} \implies \sin u = \frac{5}{\sqrt{89}}$$

(b) $\cot u$

Solution:

$$\cot u = \frac{\text{adjacent}}{\text{opposite}} \implies \cot u = \frac{b}{a} \implies \cot u = \frac{8}{5}$$

(c) $\sec v$

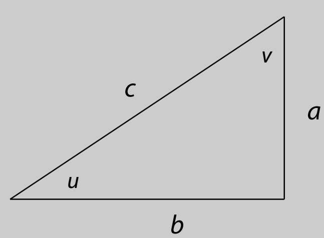
Solution:

$$\sec u = \frac{\text{hypotenuse}}{\text{adjacent}} \implies \sec u = \frac{c}{b} \implies \sec u = \frac{\sqrt{89}}{8}$$

13. Suppose $-\frac{\pi}{2} < \theta < 0$ and $\tan\theta = -3$. Evaluate

(a) $\cos\theta$

Solution: Suppose we have the following triangle:



Suppose that $u = \theta$ in the figure above. Since $\tan\theta = -3$ and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$, we can deduce that $a = -3$ and $b = 1$. Notice that the negative is associated with the side a. This is because we are looking at a triangle that is in Quadrant IV.

We can use the Pythagorean Theorem to solve for side c.

$$c^2 = (-3)^2 + 1^2 = 10 \implies c = \sqrt{10}$$

Then,

$$\cos\theta = \frac{b}{c} \implies \cos\theta = \frac{1}{\sqrt{10}}$$

(b) $\sin\theta$

Solution: From above, we know that $c = \sqrt{10}$. Thus,

$$\sin\theta = \frac{a}{c} \implies \sin\theta = -\frac{3}{\sqrt{10}}$$

14. Find the smallest number x such that $\tan e^x = 0$.

Solution: Let $y = e^x$. Then, we want to find y such that $\tan y = 0$. Thus, $y = 0, \pi, 2\pi, 3\pi, \ldots$ Substituting $y = e^x$ back into our solution for y, we see

$$e^x = 0 \text{ or } e^x = \pi \text{ or } e^x = 2\pi \ldots$$

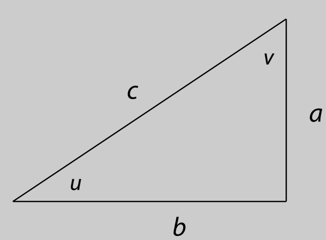
And, solving for x we find that

$$x = \ln(0) \text{ or } x = \ln(\pi) \text{ or } x = \ln(2\pi) \ldots$$

Notice that $\ln(0)$ does not exist, and since we are looking for the smallest x that makes sense, we have $x = \ln(\pi)$.

15. Suppose $-\frac{\pi}{2} < x < 0$ and $\cos x = \frac{5}{9}$. Evaluate $\sin x$ and $\tan x$.

Solution: Suppose we have the following triangle:



Suppose that $u = x$ in the figure above. Since $\cos x = \frac{5}{9}$ and $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$, we can deduce that $b = 5$ and $c = 9$. Notice that we are looking at a triangle that is in Quadrant IV, and hence, $\sin x$ and $\tan x$ will be negative.

We can use the Pythagorean Theorem to solve for side a.

$$a^2 + 5^2 = 9^2 \implies a^2 = 81 - 25 \implies a^2 = 56 \implies a = -2\sqrt{14}$$

Then,

$$\sin x = \frac{a}{c} \implies \sin x = \frac{-2\sqrt{14}}{9}.$$

And,

$$\tan x = \frac{a}{b} \implies \tan x = \frac{-2\sqrt{14}}{5}.$$

16. Find exact values for the following

(a) $\sin\left(-\frac{3\pi}{2}\right)$

Solution: Using the unit circle,

$$\sin\left(-\frac{3\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

(b) $\cos\frac{15\pi}{4}$

Solution: Using the unit circle,

$$\cos\left(\frac{15\pi}{4}\right) = \cos\left(\frac{15\pi}{4} - 2\pi\right) = \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

(c) $\cos 360045^\circ$

Solution: Using the unit circle,

$$\cos 360045^\circ = \cos\big(360045^\circ - 360^\circ \cdot 1000\big) = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

(d) $\sin 300^\circ$

Solution: Using the unit circle,

$$\sin 300^\circ = \sin(-60^\circ) = \frac{-\sqrt{3}}{2}.$$

17. Suppose an ant walks counterclockwise on the unit circle from the point $(0, 1)$ to the endpoint of the radius that forms an angle of $\frac{5\pi}{4}$ radians with the positive horizontal axis. How far has the ant walked?

Solution: The ant started at $(0, 1)$ which corresponds with the endpoint of the radius that forms an angle of $\frac{\pi}{2}$. The ant walks until he/she reaches the endpoint of the radius that forms an angle of $\frac{5\pi}{4}$. Thus, the ant walked a total of $\frac{5\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4}$ radians. The ant walked three-eighths of the circle, and since the radius of the circle is 1, the ant walked $\frac{3\pi}{4}$.

18. Let $f(x) = 3 - 5e^{2x}$.

   (a) Find the domain of f.

   Solution: Since f is an exponential function, the domain of f is all real numbers.

$$(-\infty, \infty)$$

   (b) Find the range of f.

   Solution: The range of f is equal to the domain of f^{-1}. The domain for f^{-1} is

$$(-\infty, 3).$$

   See explanation below.

   (c) Find a formula for f^{-1}.

   Solution: To find the inverse, first we set $f(x) = y$ and then solve for x.

$$\begin{aligned} y &= 3 - 5e^{2x} \\ y - 3 &= -5e^{2x} \\ \frac{y-3}{-5} &= e^{2x} \\ \ln\left(\frac{3-y}{5}\right) &= 2x && \text{(logarithm defn)} \\ \frac{\ln\left(\frac{3-y}{5}\right)}{2} &= x \end{aligned}$$

   Thus, $f^{-1}(y) = \frac{1}{2} \cdot \ln\left(\frac{3-y}{5}\right)$.

   (d) Find the domain of f^{-1}.

   Solution: Notice that f^{-1} has a logarithm and recall that the input of a logarithm must be positive. Thus,

$$\frac{3-y}{5} > 0 \implies 3 - y > 0 \implies 3 > y.$$

   The domain for f^{-1} is

$$(-\infty, 3).$$

   (e) Find the range of f^{-1}.

   Solution: The range of f^{-1} is equal to the domain of f. The domain of f is all real numbers:

$$(-\infty, \infty).$$

19. Suppose $\log_7 w = 3.1$ and $\log_7 z = 2.2$. Evaluate $\log_7\left(\frac{49w^2}{z^3}\right)$.

Solution: First, we are going to expand $\log_7\left(\frac{49w^2}{z^3}\right)$. Then, we will substitute what we know (i.e. $\log_7 w = 3.1$ and $\log_7 z = 2.2$) into the equation.

$$\begin{aligned}
\log_7\left(\frac{49w^2}{z^3}\right) &= \log_7\left(49w^2\right) - \log_7(z^3) && \text{(quotient rule)}\\
&= \log_7(7w)^2 - 3\log z && \text{(power rule)}\\
&= 2\log_7 7w - 3\log z && \text{(power rule)}\\
&= 2(\log_7 7 + \log_7 w) - 3\log z && \text{(product rule)}\\
&= 2\log_7 7 + 2\log_7 w - 3\log z\\
&= 2\cdot 1 + 2\cdot 3.1 - 3\cdot 2.2\\
&= 2 + 6.2 - 6.6\\
&= 1.6
\end{aligned}$$

20. Find all numbers x such that $e^{4x} - 9e^{2x} - 22 = 0$.

Solution: Let $y = e^{2x}$. Then, $e^{4x} - 9e^{2x} - 22 = 0$ becomes $y^2 - 9y - 22 = 0$.

$$y^2 - 9y - 22 = 0 \implies (y-11)(y+2) = 0 \implies y = 11, -2$$

Substituting e^{2x} for y, we have

$$e^{2x} = 11 \implies \ln 11 = 2x \implies x = \frac{\ln 11}{2},$$

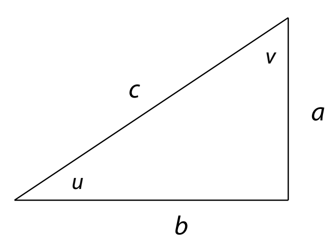
and

$$e^{2x} = -2 \implies \ln(-2) = 2x$$

But, $\ln(-2)$ does not exist. Thus, our solution is $x = \frac{\ln 11}{2}$.

21. Use the figure to the right to solve the following:

Suppose $\cos u = \frac{2}{3}$. Evaluate $\cos v$.



Solution: Notice that $\cos v = \cos(\frac{\pi}{2} - u) = \sin u$. Thus, we are looking for $\sin u$.

Since $\cos u = \dfrac{2}{3}$, we let $b = 2$ and $c = 3$. In order to find $\sin u$, we need to find a. Using the Pythagorean Theorem,

$$a^2 + 2^2 = 3^2 \implies a^2 = 9 - 4 \implies a^2 = 5 \implies a = \sqrt{5}.$$

$$\sin u = \frac{a}{c} \implies \sin u = \frac{\sqrt{5}}{3}$$

Thus, $\cos v = \dfrac{\sqrt{5}}{3}$.

22. Find a formula for the inverse of the function f defined by $f(x) = 7 - 3\log_4(2x - 1)$.

Solution: First, we set $f(x) = y$, and then we solve for x.

$$\begin{aligned} y &= 7 - 3\log_4(2x-1) \\ y - 7 &= -3\log_4(2x-1) \\ \frac{y-7}{-3} &= \log_4(2x-1) \\ 4^{\frac{7-y}{3}} &= 2x - 1 && \text{(by defn of logarithm)} \\ 4^{\frac{7-y}{3}} + 1 &= 2x \\ \frac{1}{2}\left(4^{\frac{7-y}{3}} + 1\right) &= x \end{aligned}$$

Thus, $f^{-1}(y) = \dfrac{1}{2}\left(4^{\frac{7-y}{3}} + 1\right)$.