Practice Problems - Final: Part 1 (Due Wed, May 28)

Math 1060Q – Summer 2014

Professor Hohn

1. $<$, $>$, or $=$.

   (a) $\tan(100^\circ)$ $\qquad$ $\tan(1^\circ)$

   Solution:
   $\tan(100^\circ)$ is negative and $\tan(1^\circ)$ is positive. Thus, $\tan(100^\circ) < \tan(1^\circ)$.

   (b) The solution x of $\log_{\sqrt{8}} x = \frac{8}{3}$ $\qquad$ 10

   Solution: First, let us find the solution x of $\log_{\sqrt{8}} x = \frac{8}{3}$.

$$\log_{\sqrt{8}} x = \frac{8}{3} \iff x = (\sqrt{8})^{\frac{8}{3}}$$

$$\begin{aligned} x &= (\sqrt{8})^{\frac{8}{3}} \\ &= (8^{\frac{1}{2}})^{\frac{8}{3}} \\ &= (2^{\frac{3}{2}})^{\frac{8}{3}} \\ &= 2^{\frac{3\cdot 8}{2\cdot 3}} \\ &= 2^{\frac{8}{2}} \\ &= 2^4 \\ &= 16 \end{aligned}$$

   $16 > 10$, so the solution x of $\log_{\sqrt{8}} x = \frac{8}{3} > 10$.

   (c) The period of the function $f(x) = 3\sin(\pi x - 5) + 7$ $\qquad$ The amplitude of the function $f(x) = 3\sin(\pi x - 5) + 7$

   Solution: The period of $f(x) = 3\sin(\pi x - 5) + 7$ is $p = \frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$. The amplitude of the function $f(x) = 3\sin(\pi x - 5) + 7$ is 3. Hence, the period of the function $f(x) = 3\sin(\pi x - 5) + 7 <$ the amplitude of the function $f(x) = 3\sin(\pi x - 5) + 7$.

   (d) $3\log_2 3$ $\qquad$ $2\log_5 6$

   Solution: $3\log_2 3 = x \iff 2^x = 3^3$. Similarly, $2\log_5 6 = y \iff 5^y = 6^2$. Thus, $4 < x < 5$ and $2 < y < 3$ and $3\log_2 3 > 2\log_5 6$.

   (e) The period of $f(x) = 4\tan(3x)$ $\qquad$ The period of $g(x) = 4\cos(3x)$

Solution: The period of $f(x) = 4\tan(3x)$ is $p = \frac{\pi}{b} = \frac{\pi}{3}$. The period of $g(x)$ is $p = \frac{2\pi}{b} = \frac{2\pi}{3}$. Hence, the period of $f(x) = 4\tan(3x) <$ the period of $g(x) = 4\cos(3x)$.

2. Find all solutions to $\sin(2x) + \cos x = 0$ on the interval $[0, 2\pi)$.

Solution:

$$\begin{aligned} \sin(2x) + \cos(x) &= 0 \\ 2\sin(x)\cos(x) + \cos(x) &= 0 && \text{(double angle identity)} \\ \cos(x)(2\sin(x) + 1) &= 0 \end{aligned}$$

Then, $\cos(x) = 0$ or $2\sin(x) + 1 = 0$.

$$\cos(x) = 0 \implies x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$2\sin(x) + 1 = 0 \implies \sin(x) = \frac{-1}{2} \implies x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Thus, $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

3. Give an example of a function that is neither even nor odd, and explain why it is neither.

Solution: Let $f(x) = 5x + 9$. f is even if $f(x) = f(-x)$, so let's prove that f is not even.

$$f(-x) = 5(-x) + 9 = -5x + 9 \neq 5x + 9 = f(x)$$

Since $f(x) \neq f(-x)$, f is not even. Now, we will show that f is not odd.

f is odd if $f(x) = -f(-x)$.

$$-f(-x) = -(5(-x) + 9) = -(-5x + 9) = 5x - 9 \neq 5x + 9 = f(x)$$

Since $f(x) \neq -f(-x)$, f is not odd. Hence, f is neither even nor odd.

4. Where is the function $f(x) = \frac{\sqrt{\sin x}}{x^2 - 4x + 3}$ defined on the interval $[0, 2\pi]$? Write your answer as a union of intervals.

Solution: First, let's factor f.

$$f(x) = \frac{\sqrt{\sin x}}{x^2 - 4x + 3} = \frac{\sqrt{\sin x}}{(x-3)(x-1)}$$

Now, we know that $x \neq 3, 1$. Additionally, since $\sin(x)$ is under a square root, $\sin(x) \geqslant 0$. Thus, is must also be true that $0 \leqslant x \leqslant \pi$ (x exists in the top half of the unit circle). Combining there two, we have

$$x \in [0, 1) \cup (1, 3) \cup (3, \pi].$$

5. Find an exact expression for $\sin(75°)$.

Solution:

$$\begin{aligned}\sin(75^\circ) &= \sin(30^\circ + 45^\circ)\\ &= \sin(30^\circ)\cos(45^\circ) + \cos(30^\circ)\sin(45^\circ)\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

6. Find all real numbers x such that $12x^4 + 5x^2 - 2 = 0$.

Solution: Let $y = x^2$. Then, $12x^4 + 5x^2 - 2 = 0$ becomes $12y^2 + 5y - 2 = 0$.

$$12y^2 + 5y - 2 = 0 \implies (4y-1)(3y+2) = 0 \implies 4y - 1 = 0 \text{ or } 3y + 2 = 0$$

Suppose $4y - 1 = 0$. Then,

$$4y - 1 = 0 \implies y = \frac{1}{4} \implies x^2 = \frac{1}{4} \implies x = \frac{1}{2}, \frac{-1}{2}.$$

Suppose $3y + 2 = 0$. Then,

$$3y + 2 = 0 \implies y = \frac{-2}{3} \implies x^2 = \frac{-2}{3} \implies x \text{ has no solution.}$$

7. Find the domain and range of $f(x) = \log(-x)$. What is the inverse function of $f(x)$? Find the domain and range of the inverse function of $f(x)$.

Solution: The domain of f is all possible inputs into f. Here, since the inside of a logarithm must be positive, the only acceptable inputs are negative numbers.

$$D_f : (-\infty, 0), R_f : (-\infty, \infty)$$

Now, we will find the inverse function. First, we set $y = \log(-x)$ and then we solve for x.

$$\begin{aligned}y &= \log(-x)\\ 10^y &= -x\\ -10^y &= x\end{aligned}$$

Thus, $f^{-1}(y) = -10^y$.
The domain of f^{-1} is all real numbers. The range of f^{-1} is the domain of f. Hence,

$$D_{f^{-1}} : (-\infty, \infty), R_{f^{-1}} : (-\infty, 0).$$

8. Prove the following identity

$$\sin\theta\cos\theta = \frac{\tan\theta}{1+\tan^2\theta}\,.$$

Solution: We will start on the right hand side (RHS).

$$\begin{aligned}
\frac{\tan\theta}{1+\tan^2\theta} &= \frac{\tan\theta}{\sec^2\theta} && \text{(identity)}\\
&= \frac{\tan\theta}{\frac{1}{\cos^2\theta}} && \text{(defn of sec)}\\
&= \frac{\tan\theta}{1} \div \frac{1}{\cos^2\theta}\\
&= \frac{\tan\theta}{1} \cdot \frac{\cos^2\theta}{1}\\
&= \frac{\sin\theta}{\cos\theta} \cdot \cos^2\theta && \text{(defn of tan)}\\
&= \sin\theta\cos\theta
\end{aligned}$$

Thus, $\sin\theta\cos\theta = \dfrac{\tan\theta}{1+\tan^2\theta}$.

9. Find the linear function, $y = mx + b$, that passes through the vertices of $y = x^2 + 4x$ and $y = 2(x+1)^2$.

Solution: First, we will find the vertices of the parabolas. Then, we will find the line through these points. The vertex of $y = x^2 + 4x$ can be found by completing the square.

$$\begin{aligned}
y &= x^2 + 4x\\
&= (x^2 + 4x + 4 - 4)\\
&= (x^2 + 4x + 4) - 4\\
&= (x+2)^2 - 4
\end{aligned}$$

Thus, the vertex is $(-2,-4)$. The vertex of $y = 2(x+1)^2$ is $(-1,0)$ (the square was already completed for us). Now, we need to find the line that passes through these two points. First, we will find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \implies m = \frac{0-(-4)}{-1-(-2)} \implies m = \frac{4}{1}.$$

Then, an equation in point-slope form that passes through those point is

$$y - 0 = 4(x - (-1)) \implies y = 4(x+1) \implies y = 4x + 4.$$

10. A population of 8 frogs increases at an annual rate of 50% a year. How many frogs will there be in 4 years?

Solution: We are looking at an exponential growth problem. Thus, we are looking for a function of the form $f(x) + cb^x$. We know $f(0) = 8$ and $f(1) = 8 + 8 \cdot 0.5 = 12$. Thus,

$$f(0) = cb^0 = c = 8 \text{ and } f(1) = cb^1 = 8b^1 = 12 \implies b = \frac{12}{8} \implies b = \frac{3}{2}.$$

The function describing the growth is $f(x) = 8\left(\frac{3}{2}\right)^x$. At the end of year 4, we would have

$$f(4) = 8\left(\frac{3}{2}\right)^4 = 8 \cdot \frac{3^4}{2^4} = 8 \cdot \frac{81}{16} = \frac{81}{2} = 40.5.$$

We can do this calculation without worrying about exponential growth if we desire. At the end of year 1, we would have $8 + 8 \cdot 0.5 = 8 + 4 = 12$. At the end of year 2, $12 + 12 \cdot 0.5 = 12 + 6 = 18$. At the end of year 3, $18 + 18 \cdot 0.5 = 18 + 9 = 27$. Finally, after the end of year 4, $27 + 27 \cdot 0.5 = 27 + 13.5 = 40.5$ frogs.

11. Suppose $\sin u = \frac{3}{7}$. Evaluate $\cos(2u)$.

Solution: Recall that $\cos(2u) = 1 - 2\sin^2 u$. Then,

$$\cos(2u) = 1 - 2 \cdot \left(\frac{3}{7}\right)^2 \implies \cos(2u) = 1 - \frac{2 \cdot 9}{49} \implies \cos(2u) = \frac{49 - 18}{49} \implies \cos(2u) = \frac{31}{49}.$$

12. Suppose $9^x = 4$. Evaluate $(\frac{1}{27})^{2x}$.

Solution: Notice that

$$(\frac{1}{27})^{2x} = (\frac{1}{3^3})^{2x} = (3^{-3})^{2x} = 3^{-6x} = (3^{2x})^{-3}.$$

Also, notice that

$$9^x = (3^2)^x = 3^{2x} = 4.$$

Thus,

$$\left(\frac{1}{27}\right)^{2x} = \left(3^{2x}\right)^{-3} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

.

13. The function f is defined by $f(-3) = 8$, $f(1) = 4$, and $f(4) = -8$. Make a table for $g(x)$ where $g(x) = 2f(-5x + 1) - 3$.

Solution: First, let's draw a table for f.

x	$f(x)$
-3	8
1	4
4	-8

We need to find the values that we can input into g for our x column. Since f only allows the inputs $-3, 1, 4$, g must have inputs such that $-5x+1=-3$, $-5x+1=1$ and $-5x+1=4$. So, the table for $g(x)$ would be:

x	$g(x) = 2f(-5x+1) - 3$
$\frac{4}{5}$	$g(\frac{4}{5}) = 2f(-5(\frac{4}{5})+1) - 3 = 2f(-3) - 3 = 2 \cdot 8 - 3 = 13$
0	$g(0) = 2f(-5(0)+1) - 3 = 2f(1) - 3 = 2 \cdot 4 - 3 = 5$
$\frac{-3}{5}$	$g(\frac{-3}{5}) = 2f(-5(\frac{-3}{5})+1) - 3 = 2f(4) - 3 = 2 \cdot -8 - 3 = -19$

14. What is $\sin^{-1}(\sin(\frac{3\pi}{4}))$?

Solution: First, let us find $\sin(\frac{3\pi}{4})$.

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \text{(in Quadrant II)}$$

Now, we need to find $\sin^{-1}(\frac{\sqrt{2}}{2})$.

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \qquad \left(\text{since } -\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}\right)$$

Thus, $\sin^{-1}(\sin(\frac{3\pi}{4})) = \frac{\pi}{2}$.

15. What is the minimum value of the function f defined by $f(x) = 9x^2 + 30x + 18$?

Solution: To find the minimum value of f, we must first find the vertex of f which we will do by completing the square.

$$\begin{aligned}
f(x) &= 9x^2 + 30x + 18 \\
&= 9\left(x^2 + \frac{30}{9} \qquad\right) + 18 \\
&= 9\left(x^2 + \frac{10}{3} \qquad\right) + 18 \\
&= 9\left(x^2 + \frac{10}{3} + \left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right) + 18 \\
&= 9\left(x^2 + \frac{10}{3} + \left(\frac{5}{3}\right)^2\right) - \left(\frac{5}{3}\right)^2 \cdot 9 + 18 \\
&= 9\left(x^2 + \frac{10}{3} + \left(\frac{5}{3}\right)^2\right) - \left(\frac{25}{9}\right) \cdot 9 + 18 \\
&= 9\left(x^2 + \frac{10}{3} + \left(\frac{5}{3}\right)^2\right) - 25 + 18 \\
&= 9\left(x^2 + \frac{5}{3}\right)^2 - 7
\end{aligned}$$

Thus, the vertex is $(-\frac{5}{3}, -7)$. Hence, the minimum of f is -7.

16. Find an exact expression for $\sin(\frac{\pi}{8})$.

Solution: We can find $\sin(\frac{\pi}{8})$ by using the half-angle formula. That is, $\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\frac{\pi}{4}}{2}\right)$.

$$\begin{aligned}
\sin\left(\frac{\pi}{8}\right) &= \sin\left(\frac{\frac{\pi}{4}}{2}\right) \\
&= \sqrt{\frac{1-\cos(\frac{\pi}{4})}{2}} && \text{(square root positive since } \sin(\frac{\pi}{4}) \geqslant 0) \\
&= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}
\end{aligned}$$

Thus, $\sin(\frac{\pi}{8}) = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$.