Practice Problems - Final: Part 2 (Due Thurs, May 29)

Math 1060Q – Summer 2014
Professor Hohn

1. Write $2^8 \dfrac{4^{77}}{16^{28}}$ as a power of 4.

Solution: Since $4 = 2^2$ and $16 = 4^2$, then $2^8 = \left(2^2\right)^4 = 4^4$ and $16^{28} = \left(4^2\right)^{28} = 4^{56}$. So,

$$2^8 \frac{4^{77}}{16^{28}} = 4^4 \frac{4^{77}}{4^{56}} = 4^{77+4-56} = 4^{25}.$$

2. Find the smallest possible positive number x such that $16 \sin^4 x - 16 \sin^2 x + 3 = 0$.

Solution: Let's set $y = \sin^2 x$, then we have

$$16y^2 - 16y + 3 = 0.$$

If you can factor this then you will see

$$16y^2 - 16y + 3 = 0 \implies (4y-3)(4y-1) = 0 \implies y = \frac{3}{4}, \frac{1}{4}.$$

Using the quadratic formula, the roots of $16y^2 - 16y + 3$ are

$$y = \frac{16 \pm \sqrt{(-16)^2 - 4(16)(3)}}{2(16)} = \frac{1}{2} \pm \frac{1}{4} = \frac{1}{4}, \frac{3}{4}$$

Therefore, we are looking for the smallest positive x such that either

$$\sin^2 x = \frac{1}{4} \implies \sin x = \pm\frac{1}{2} \implies x = \frac{\pi}{6} + 2\pi k \text{ or } x = -\frac{\pi}{6} + 2\pi k.$$

or

$$\sin^2 x = \frac{3}{4} \implies \sin x = \pm\frac{\sqrt{3}}{2} \implies \text{ or } x = \frac{\pi}{3} + 2\pi k \text{ or } x = -\frac{\pi}{3} + 2\pi k.$$

Of all these solutions, $x = \frac{\pi}{6}$ is the smallest positive one.

3. Give an example of an odd function whose domain is the real numbers and whose range is $\{-\pi^2, 0, \pi^2\}$.

Solution: We will construct a function whose domain is all real numbers and whose range is $\{-\pi^2, 0, \pi^2\}$.

$$f(x) = \begin{cases} -\pi^2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \pi^2 & \text{if } x > 0 \end{cases}$$

Now, we will show that f is odd.

$$-f(-x) = \begin{cases} \pi^2 & \text{if } -x < 0 \\ 0 & \text{if } x = 0 \\ -\pi^2 & \text{if } -x > 0 \end{cases}$$

which is equal to $f(x)$.

4. Let $f(x) = \dfrac{x^4 - 2x^2 - 35}{2x^4 - 8}$. Find the vertical asymptotes and end behavior of $f(x)$. What are the zeros of f?

Solution: Factoring the numerator, we get

$$x^4 - 2x^2 - 35 = (x^2 - 7)(x^2 + 5) = (x + \sqrt{7})(x - \sqrt{7})(x^2 + 5).$$

Factoring the denominator, we get

$$2x^4 - 8 = 2(x^4 - 4) = 2(x^2 + 2)(x^2 - 2) = 2(x^2 + 2)(x + \sqrt{2})(x - \sqrt{2})$$

Hence, rewriting the rational function, we find

$$\frac{x^4 - 2x^2 - 35}{2x^4 - 8} = \frac{(x + \sqrt{7})(x - \sqrt{7})(x^2 + 5)}{2(x^2 + 2)(x + \sqrt{2})(x - \sqrt{2})}$$

A vertical asymptote occurs when there is a zero in the denominator that is not a hole; these are $x = \sqrt{2}$ and $x = -\sqrt{2}$. The zeros of the function happen when there is a zero in the numerator which is not a hole; these are $x = \sqrt{7}$ and $x = -\sqrt{7}$. Finally, the end behavior acts the same as the ratio of the largest powered term in the numerator over the largest powered term in the denominator:

$$\lim_{x \to \infty} \frac{x^4 - 2x^2 - 35}{2x^4 - 8} = \lim_{x \to \infty} \frac{x^4}{2x^4} = \lim_{x \to \infty} \frac{1}{2}$$

and

$$\lim_{x \to -\infty} \frac{x^4 - 2x^2 - 35}{2x^4 - 8} = \lim_{x \to -\infty} \frac{x^4}{2x^4} = \lim_{x \to -\infty} \frac{1}{2}.$$

5. Calculate $\log(\frac{1}{2}) + \log(\frac{2}{3}) + ... + \log(\frac{99}{100})$.

Solution: Let's write out the first few terms to discover the pattern:

$$\begin{aligned}\log(1/2)+\log(2/3)+\log(3/4) &= \big(\log(1)-\log(2)\big)+\big(\log(2)-\log(3)\big)+\big(\log(3)-\log(4)\big)\\ &= \log(1)\underbrace{-\log(2)+\log(2)}_{=0}\underbrace{-\log(3)+\log(3)}_{=0}-\log(4)=\log(1)-\log(4).\end{aligned}$$

If we kept writing out terms, we'd see that all we'd be left with after canceling terms are $\log(1)-\log(100)=-\log(100)$ (since $\log(1)=0$). Also, $\log(100)=\log(10^2)=2\log(10)=2$. So, the answer is -2.

6. Evaluate $\cos(\tan^{-1}5)$.

Solution: Let $\theta=\tan^{-1}(5)$. Then since $\tan(\theta)=5$ which is the ratio of the "opposite over adjacent", we can picture a right triangle with opposite side having length 5 and adjacent side having length 1. In this case, the hypotenuse has length $\sqrt{5^2+1^2}=\sqrt{26}$. Now, since $\cos(\theta)$ is the ratio of "adjacent over hypotenuse", we have that $\cos(\theta)=1/\sqrt{26}$.

7. High tide at La Jolla Cove occurs at 5 am and is 6.5 ft. Low tide occurs at 11 am and is -0.5 ft. A simple model for such tides could be a cosine function of the form $f(x)=a\cos(bx+c)+d$. Determine the values for $a>0$, b, c, and d for $f(x)$ where x represents the number of hours since midnight. Sketch $f(x)$.

Solution:

First, let us find the amplitude, a. Notice that high tide is 6.5ft and low tide is -0.5ft. Hence,

$$a=\frac{\text{max-min}}{2}\implies a=\frac{6.5-(-0.5)}{2}\implies a=\frac{7}{2}.$$

Now, let us find the vertical shift d:

$$d=\max-a\implies d=6.5-3.5\implies d=3.$$

High tide and low tide occur every 12 hours. Hence, the period is 12. We will use this information to find b:

$$p=\frac{2\pi}{b}\implies 12=\frac{2\pi}{b}\implies b=\frac{2\pi}{12}\implies b=\frac{\pi}{6}.$$

The equation we have so far is

$$f(x)=3.5\cos\left(\frac{\pi}{6}x+c\right)+3.$$

Now, we will find c by using the information that $f(5)=6.5$.

$$f(5)=3.5\cos\left(\frac{5\pi}{6}+c\right)+3\implies 6.5=3.5\cos\left(\frac{5\pi}{6}+c\right)+3\implies 1=\cos\left(\frac{5\pi}{6}+c\right)$$

Solving for c,

$$\cos^{-1}(1)=\frac{5\pi}{6}+c\implies 0=\frac{5\pi}{6}+c\implies c=-\frac{5\pi}{6}$$

Thus, our equation is

$$f(x)=3.5\cos\left(\frac{\pi}{6}x-\frac{5\pi}{6}\right)+3.$$

Listing out the variables: $a=3.5, b=\pi/6, c=-5\pi/6$, and $d=3$.

8. Simplify each of the following expressions.

   (a) $\sin^{-1}(\cos(\frac{5\pi}{6}))$

   Solution: We have $\cos(5\pi/6) = -\sqrt{3}/2$. Then $\sin^{-1}(-\sqrt{3}/2) = \theta$ where $\sin(\theta) = -\frac{\sqrt{3}}{2} \implies \theta = -\frac{\pi}{3}$.

   (b) $\sin(\cos^{-1} x)$

   Solution: If $\theta = \cos^{-1} x$ then $\cos\theta = x$; which as a ratio of side of a right triangle means that the "adjacent over hypotenuse" equals x, so let's set the adjacent side length to x and the hypotenuse length equal to 1. Then, $\sin(\theta)$ will be the ratio of "opposite over hypotenuse". The opposite leg will have length $\sqrt{1^2 - x^2} = \sqrt{1 - x^2}$; therefore $\sin(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1 - x^2}$.

9. Solve for x in the following equations.

   (a) $\log_4 x + \log_4(x - 3) = 1$

   Solution: First $\log_4 x + \log_4(x - 3) = \log_4(x \cdot (x - 3)) = \log_4(x^2 - 3x)$. This reduces us to solving $\log_4(x^2 - 3x) = 1$. We have

   $$\log_4(x^2 - 3x) = 1 \implies 4^1 = x^2 - 3x \implies 0 = x^2 - 3x - 4 \implies 0 = (x - 4)(x + 1).$$

   Then, $x = 4, -1$. Now, we must check our solutions to make sure they make sense. Notice that $\log(-1)$ is undefined. Thus, $x \neq -1$ and our solution is $x = 4$.

   (b) $e^x + 2e^{-x} = 3$

   Solution: First multiply through by e^x to get $e^{2x} + 2 = 3e^x$. Now, rearranging the terms yields $(e^x)^2 - 3e^x + 2 = 0$. Let $y = e^x$ and this equation is $y^2 - 3y + 2 = 0$. This factors to $(y - 2)(y - 1) = 0$ implying the solutions are $y = 2$ and $y = 1$. Therefore, we are looking for the solutions to $e^x = 2 \implies x = \ln 2$ and $e^x = 1 \implies x = 0$. Both solutions make sense in the original equation, so the solutions are $x = \ln 2$ and $x = 0$.

10. Each year the local country club sponsors a tennis tournament. Play starts with 128 participants. During each round, half of the players are eliminated. How many players remain after 5 rounds?

    Solution: After the first round we have 128(.5) players left. After the second round we will have half of these, leaving $128(.5)(.5) = 128(.5)^2$ players. Continuing this way, we see that after 5 rounds we will have $128(.5)^5$ players, which is $128(.5)^5 = 128/32 = 4$.

11. Show that $\sin^2(2x) = 4(\sin^2 x - \sin^4 x)$. Hint: Start on the left-hand side and use the double angle identity.

Solution: We will start on the left hand side and show it is equal to the right hand side.

$$\begin{aligned}\sin^2(2x) &= [\sin(2x)]^2 \\ &= [2\sin x\cos x]^2 && \text{(from the double angle formula)} \\ &= 4\sin^2 x\cos^2 x \\ &= 4\sin^2 x(1-\sin^2 x) && \text{(from the identity } \cos^2+\sin^2=1) \\ &= 4\sin^2 x - 4\sin^4 x\end{aligned}$$

Therefore, the left and right hand sides are equal.

12. Sketch the graph of the function $4\sin(2x+1)+5$ on the interval $[-3\pi, 3\pi]$.

Solution: Done in class.

13. Let $f(x) = \dfrac{6x+1}{5x-9}$.

(a) Find the domain of f.

Solution: The only trouble with values x in $\frac{6x+1}{5x-9}$ is when $5x-9=0 \implies x = 9/5$. So, the domain is

$$(-\infty, \frac{9}{5}) \cup (\frac{9}{5}, \infty).$$

(b) Find the range of f.

Solution: Do parts c and d first. Using part d, the domain of f^{-1} is the range of f, so the range of f is

$$(-\infty, \frac{6}{5}) \cup (\frac{6}{5}, \infty).$$

(c) Find a formula for f^{-1}.

Solution: Let y be such that $\dfrac{6x+1}{5x-9} = y$. Then,

$$6x+1 = 5xy - 9y \implies x(6-5y) = -9y-1 \implies x = \frac{-9y-1}{-5y+6}.$$

Therefore, $f^{-1}(y) = \dfrac{-9y-1}{-5y+6}$.

(d) Find the domain of f^{-1}.

Solution: The only trouble with values of y in $\dfrac{-9y-1}{-5y+6}$ is when $-5y+6=0 \implies y = 6/5$. So the domain of $f^{-1}(y)$ is

$$(-\infty, \frac{6}{5}) \cup (\frac{6}{5}, \infty).$$

(e) Find the range of f^{-1}.

Solution: The range of f^{-1} is the domain of f, which from part a gives

$$(-\infty, \frac{9}{5}) \cup (\frac{9}{5}, \infty).$$