Name: ______________________________

Score: _______ /15

Practice Problems - Final: Part 3 (Due Fri, May 30)

Math 1060Q – Summer 2014

Professor Hohn

1. Simplify the expression $\left(\frac{(3t^9w^{-5})^4}{(t^{-3}w^7)^5}\right)^{-2}$.

Solution:

$$\begin{aligned}\left(\frac{(3t^9w^{-5})^4}{(t^{-3}w^7)^5}\right)^{-2} &= \left(\frac{3^4t^{36}w^{-20}}{t^{-15}w^{35}}\right)^{-2}\\ &= \frac{3^{-8}t^{-72}w^{40}}{t^{30}w^{-70}}\\ &= 3^{-8}t^{-72-30}w^{40+70}\\ &= 3^{-8}t^{-102}w^{110}\\ &= \frac{w^{110}}{3^8t^{102}}\end{aligned}$$

2. Suppose you go to the fair and decide to ride the Ferris Wheel. The Ferris Wheel has a 30 meter diameter and turns 3 revolutions per minute with its lowest point 1 meter off the ground. Assume your height h above the ground is a function of the form $h(x) = a\cos(bx + c) + d$, where $x = 0$ represents the lowest point on the wheel and x is measured in seconds. Find the values of $a > 0$, $b > 0$, c, and d.

Solution: First, let us find the amplitude a. Notice that the Ferris Wheel is 30 meters in diameter and is also 1 meter off the ground. In total, the Ferris Wheel has a maximum height of 31 and a minimum height of 1.

$$a = \frac{\text{max - min}}{2} \implies a = \frac{31-1}{2} \implies a = \frac{30}{2} \implies a = 15$$

Now, let us find the vertical shift d.

$$d = \max - a \implies d = 31 - 15 \implies d = 16$$

The period of the Ferris Wheel is 20 seconds (fast and furious!). Thus,

$$p = \frac{2\pi}{b} \implies 20 = \frac{2\pi}{b} \implies b = \frac{2\pi}{20} \implies b = \frac{\pi}{10}.$$

Our equation so far is

$$f(x) = 15\cos\left(\frac{\pi}{10}x + c\right) + 16.$$

We can use the point $f(0) = 1$ to help us solve for c.

$$f(0) = 15\cos\left(\frac{\pi}{10}\cdot 0 + c\right)+16 \implies 1 = 15\cos(c)+16 \implies -15 = 15\cos(c) \implies -1 = \cos(c)$$

So, $c = \pi$. Then,

$$f(x) = 15\cos\left(\frac{\pi}{10}x + \pi\right) + 16.$$

Listing the variables: $a = 15, b = \frac{\pi}{10}, c = \pi$, and $d = 16$.

3. Show that $2 - \log x = \log\left(\frac{100}{x}\right)$ for every positive x.

Solution: Using logarithm rules:

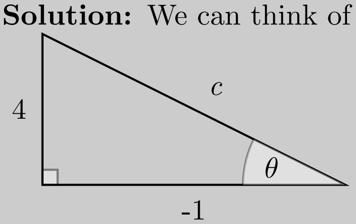
$$\begin{aligned}\log\left(\frac{100}{x}\right) &= \log(100) - \log(x) && \text{(quotient rule)}\\ &= 2 - \log(x) && \text{(since } 10^2 = 100\text{)}\end{aligned}$$

Thus, $\log\left(\frac{100}{x}\right) = 2 - \log x$.

4. Suppose $\frac{\pi}{2} < \theta < \pi$ and $\tan\theta = -4$. Evaluate

(a) $\cos\theta$

Solution: We can think of $\tan\theta$ as the ratio "opposite over adjacent."



Because we are in Quadrant II, the $\cos\theta$ will be negative and the $\sin\theta$ will be positive. The $\cos\theta$ can be thought of as the ratio "adjacent over hypotenuse" so we will find the hypotenuse first.

$$c^2 = 4^2 + (-1)^2 \implies c^2 = 16 + 1 \implies c = \sqrt{17}$$

Then, $\cos\theta = \frac{-1}{\sqrt{17}}$.

(b) $\sin\theta$

Solution: $\sin\theta = \dfrac{4}{\sqrt{17}}$.

5. Find exact values for the following
   (a) $\sin(-\frac{3\pi}{2})$

   Solution: $\sin(-\frac{3\pi}{2}) = 1$

   (b) $\cos\frac{15\pi}{4}$

   Solution: The angle $\frac{15\pi}{4}$ is in Quadrant IV, so $\cos\frac{15\pi}{4} = \dfrac{\sqrt{2}}{2}$.

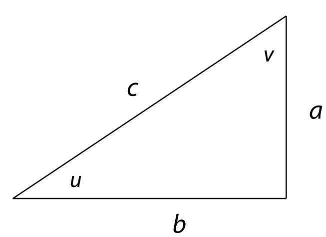
   (c) $\cos 360045^\circ$

   Solution: The angle 360045° is in Quadrant I, so $\cos 360045^\circ = \dfrac{\sqrt{2}}{2}$.

   (d) $\sin 300^\circ$

   Solution: The angle 300° is in Quadrant IV, so $\cos 300^\circ = \dfrac{-\sqrt{3}}{2}$.

6. Use the figure below to solve the following:



Suppose $a = 3$ and $c = 8$. Evaluate

   (a) $\cos v$

   Solution: The $\cos v = \dfrac{a}{c}$ which means $\cos v = \dfrac{3}{8}$.

   (b) $\sin v$

   Solution: The $\sin v = \dfrac{b}{c}$ which means we will find b first.

$$8^2 = 3^2 + b^2 \implies 64 = 9 + b^2 \implies b^2 = 55 \implies b = \sqrt{55}$$

   Thus, $\sin v = \dfrac{\sqrt{55}}{8}$.

   (c) $\tan v$

Solution: The $\tan v = \dfrac{b}{a}$ which means $\tan v = \dfrac{\sqrt{55}}{3}$.

7. Evaluate $\cos(\cos^{-1} \frac{2}{5})$.

Solution: $\cos(\cos^{-1} \frac{2}{5})$ means $\cos(\theta)$ where $\theta = \cos^{-1} \frac{2}{5}$. Then, we know $\cos\theta = \frac{2}{5}$. Since we are trying to find $\cos\theta$, we are done.

$$\cos\left(\cos^{-1}\frac{2}{5}\right) = \frac{2}{5}$$

8. Find a number b such that $\cos x + \sin x = b\sin(x + \frac{\pi}{4})$.

Solution: We will start with the right hand side.

$$\begin{aligned} b\sin\left(x + \frac{\pi}{4}\right) &= b\left(\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}\right) \\ &= b\left(\sin x\frac{\sqrt{2}}{2} + \cos x\frac{\sqrt{2}}{2}\right) \\ &= b\frac{\sqrt{2}}{2}\left(\sin x + \cos x\right) \end{aligned}$$

We want to find a b such that the right hand side looks like the left hand side. Thus,

$$1 = b\frac{\sqrt{2}}{2} \implies b = \frac{2}{\sqrt{2}} \implies b = \sqrt{2}.$$

9. Show that $\dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$ for every x that is not an integer multiple of π.
Hint: Start on the left-hand side and multiply both the numerator and the denominator by $1+\cos x$.

Solution: We will start on the left hand side and show that it is equal to the right hand side.

$$\begin{aligned} \frac{\sin x}{1-\cos x} &= \frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} && \text{(from the hint)} \\ &= \frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)} \\ &= \frac{\sin x(1+\cos x)}{(1-\cos^2 x)} \\ &= \frac{\sin x(1+\cos x)}{\sin^2 x} && \text{(from } \cos^2 + \sin^2 = 1\text{)} \\ &= \frac{1+\cos x}{\sin x} && \text{(cancelling } \sin^x\text{)} \end{aligned}$$

And, $\dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$.

10. For $f(x) = \dfrac{x-1}{x^2+1}$ and $g(x) = \dfrac{x+3}{x+4}$, find the formulas for the following.

Simplify your results as much as possible.

(a) $f \circ g$

Solution:

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f\left(\frac{x+3}{x+4}\right) \\
&= \frac{\frac{x+3}{x+4} - 1}{\left(\frac{x+3}{x+4}\right)^2 + 1} \\
&= \frac{\frac{x+3}{x+4} - \frac{x+4}{x+4}}{\frac{(x+3)^2}{(x+4)^2} + \frac{(x+4)^2}{(x+4)^2}} \\
&= \frac{\frac{x+3-(x+4)}{x+4}}{\frac{(x+3)^2+(x+4)^2}{(x+4)^2}} \\
&= \frac{\frac{-1}{x+4}}{\frac{(x+3)^2+(x+4)^2}{(x+4)^2}} \\
&= \frac{-1}{x+4} \div \frac{(x+3)^2+(x+4)^2}{(x+4)^2} \\
&= \frac{-1}{x+4} \cdot \frac{(x+4)^2}{(x+3)^2+(x+4)^2} \\
&= \frac{-1(x+4)}{(x+3)^2+(x+4)^2} \\
&= \frac{-x-4}{x^2+6x+9+x^2+8x+16} \\
&= \frac{-x-4}{2x^2+14x+25}
\end{aligned}$$

(b) $g \circ f$

Solution:

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g\left(\frac{x-1}{x^2+1}\right) \\ &= \frac{\frac{x-1}{x^2+1}+3}{\frac{x-1}{x^2+1}+4} \\ &= \frac{\frac{x-1}{x^2+1}+\frac{3(x^2+1)}{x^2+1}}{\frac{x-1}{x^2+1}+\frac{4(x^2+1)}{x^2+1}} \\ &= \frac{\frac{x-1+3(x^2+1)}{x^2+1}}{\frac{x-1+4(x^2+1)}{x^2+1}} \\ &= \frac{x-1+3(x^2+1)}{x-1+4(x^2+1} \\ &= \frac{x-1+3x^2+3)}{x-1+4x^2+4} \\ &= \frac{3x^2+x+2}{4x^2+x+3} \end{aligned}$$

11. Evaluate the expression $\sin\left[\sec^{-1}(\frac{5}{3}) + \tan^{-1}(\frac{3}{4})\right]$.

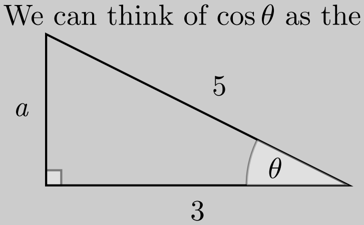
Solution: Let $\sec^{-1}(\frac{5}{3}) = \theta$ and let $\tan^{-1}(\frac{3}{4}) = \phi$. Then, we want to find $\sin(\theta + \phi)$.

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Then, we need to find the $\cos\theta$ and $\sin\theta$ as well as the $\cos\phi$ and $\sin\phi$. Recall that $\sec^{-1}(\frac{5}{3}) = \theta$. This means

$$\sec\theta = \frac{5}{3} \implies \frac{1}{\cos\theta} = \frac{5}{3} \implies \cos\theta = \frac{3}{5}.$$

We can think of $\cos\theta$ as the ratio "adjacent over hypotenuse."

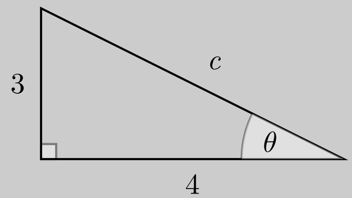


Notice that we have a $3-4-5$ triangle and $a = 4$.

$$\cos\theta = \frac{3}{5} \implies \sin\theta = \frac{4}{5}$$

Now, we will solve for $\cos\phi$ and $\sin\phi$.

We can think of $\tan\theta$ as the ratio "opposite over adjacent."



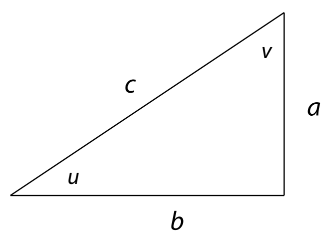
Notice that we have another $3-4-5$ triangle and $c = 5$. Thus,

$$\cos\phi = \frac{4}{5} \text{ and } \sin\phi = \frac{3}{5}.$$

We substitute our information into the equation for $\sin(\theta+\phi)$.

$$\begin{aligned}\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\ &= \frac{4}{5}\cdot\frac{4}{5} + \frac{3}{5}\cdot\frac{3}{5} \\ &= \frac{16}{25} + \frac{9}{25} \\ &= 1\end{aligned}$$

12. Use the figure below to solve the following:



Suppose $b = 3$ and $\sin v = \frac{1}{3}$.
Evaluate a.

Solution: Notice that we are dealing with a situation where we need to work with similar triangles. One triangle, call it Triangle 1, has $b = 3$, and we want to find a. The other triangle, call it Triangle 2, that is similar (the triangles have the same angles) has $b = 1$ and $c = 3$. Then, Triangle 2's missing side is

$$a^2 + 1^2 = 3^2 \implies a^2 = 9 - 1 \implies a = \sqrt{8}$$

Since Triangle 1 and Triangle 2 are similar, the sides of Triangle 1 proportional to the sides of Triangle 2. Thus, we can think of Triangle 1 as Triangle 2 scaled by 3. So, $a = 3\sqrt{8}$.

13. Let $g(x)$ be of the form $g(x) = a\cos(bx+c)+d$. Find the values for a, b, c, and d with $a > 0$, $b > 0$, and $0 \leqslant c \leqslant \pi$ so that g has range $[-3, 4]$, $g(0) = 2$, and g has period 5.

Solution: First, let us find the amplitude, a.

$$a = \frac{\text{max-min}}{2} \implies a = \frac{4-(-3)}{2} \implies a = \frac{7}{2}.$$

Now, let us find the vertical shift d:

$$d = \max - a \implies d = 4 - 3.5 \implies d = 0.5.$$

We are told that the period of g is 5. We will use this information to find b:

$$p = \frac{2\pi}{b} \implies 5 = \frac{2\pi}{b} \implies b = \frac{2\pi}{5}.$$

The equation we have so far is

$$g(x) = 3.5\cos\left(\frac{2\pi}{5}x + c\right) + 0.5.$$

Now, we will find c by using the information that $g(0) = 2$.

$$g(0) = 3.5\cos\left(\frac{2\pi}{5}0 + c\right) + 0.5 \implies 2 = 3.5\cos(c) + 0.5 \implies \frac{1.5}{3.5} = \cos(c)$$

Solving for c,

$$c = \cos^{-1}\left(\frac{1.5}{3.5}\right) \implies c = \cos^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{2}}\right) \implies c = \cos^{-1}\left(\frac{3}{7}\right)$$

Thus, our equation is

$$g(x) = 3.5\cos\left(\frac{2\pi}{5}x + \cos^{-1}\left(\frac{3}{7}\right)\right) + 0.5.$$

Listing out the variables: $a = 3.5, b = \dfrac{2\pi}{5}, c = \cos^{-1}\left(\dfrac{3}{7}\right)$, and $d = 0.5$.