Graded by: ____________________ Score: ________ ID: ________________________

WORKSHEET 6 - DUE 10/12

MATH 2110Q – Fall 2015
Professor Hohn

You must show all of your work for full credit. Please circle/box your answers or write a brief sentence indicating your answer.

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1. Find the equation of the tangent plane to the surface at the given point.

   (a) $z = 3y^2 - 2x^2 + x,\ (2, -1, -3)$

   Solution: The equation of a tangent plane to a surface is

   $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   Thus, we need to find $f_x(x, y)$ and $f_y(x, y)$.

   $$f_x(x, y) = -4x + 1 \implies f_x(2, -1) = -4(2) + 1 = -7$$

   $$f_y(x, y) = 6y \implies f_y(2, -1) = 6(-1) = -6$$

   Then, the equation of a tangent plane to our surface is

   $$z = -3 - 7(x - 2) - 6(y + 1).$$

   (b) $z = x\sin(x + y),\ (-1, 1, 0)$

   Solution: The equation of a tangent plane to a surface is

   $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   Thus, we need to find $f_x(x, y)$ and $f_y(x, y)$.

   $$f_x(x, y) = x\cos(x + y) + \sin(x + y) \implies f_x(-1, 1) = -1\cos(0) + \sin(0) = -1$$

   $$f_y(x, y) = x\cos x + y \implies f_y(-1, 1) = -1\cos(0) = -1$$

   Then, the equation of a tangent plane to our surface is

   $$z = 0 - (x + 1) - (y - 1).$$

2. The temperature at point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in $^\circ C$ and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 1)$ in the direction toward the point $Q(3, -3, 3)$.

Solution: We want to find the directional derivative of T in the direction $\vec{PQ}$. Let $\hat{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|}$. Then, we want

$$D_{\hat{u}}T(2, -1, 1) = \nabla T(2, -1, 1) \cdot \hat{u}.$$

So,

$$\nabla T(x, y, z) = \langle -2x \cdot 200e^{-x^2-3y^2-9z^2}, -6y \cdot 200e^{-x^2-3y^2-9z^2}, -18z \cdot 200e^{-x^2-3y^2-9z^2} \rangle$$

At the point $(2, -1, 1)$,

$$\nabla T(2, -1, 1) = \langle -800e^{-16}, 1200e^{-16}, -1800e^{-16} \rangle.$$

Now, we need to calculate $\hat{u}$.

$$\begin{aligned} \vec{PQ} &= \langle 3 - 2, -3 - (-1), 3 - 1 \rangle \\ &= \langle 1, -2, 2 \rangle \end{aligned}$$

Then,

$$\begin{aligned} \hat{u} &= \frac{\langle 1, -2, 2 \rangle}{\sqrt{1^2 + (-2)^2 + 2^2}} \\ &= \langle 1/3, -2/3, 2/3 \rangle \end{aligned}$$

Then,

$$\begin{aligned} \nabla T(2, -1, -1) \cdot \hat{u} &= \langle -800e^{-16}, 1200e^{-16}, -1800e^{-16} \rangle \cdot \langle 1/3, -2/3, 2/3 \rangle \\ &= \frac{-800e^{-16}}{3} + \frac{-2400e^{-16}}{3} + \frac{-3600e^{-16}}{3} \\ &= \frac{-6800e^{-16}}{3} \end{aligned}$$

(b) In which direction does the temperature increase the fastest at P?

Solution: Recall that the greatest rate of change occurs in the direction of the gradient at the P we are looking at. Hence,

$$\nabla T(2, -1, 1) = \langle -800e^{-16}, 1200e^{-16}, -1800e^{-16} \rangle$$

(c) Find the maximum rate of increase at P.

Solution: Recall that the maximum rate of increase is the magnitude of the gradient vector at our point. Hence,

$$\begin{aligned} \|\nabla T(2,-1,1)\| &= \left\|\langle -800e^{-16}, 1200e^{-16}, -1800e^{-16}\rangle\right\| \\ &= \left|200e^{-16}\right| \|\langle 4,6,9\rangle\| \\ &= 200e^{-16}\sqrt{4^2+6^2+9^2} \\ &= 200e^{-16}\sqrt{16+36+81} \\ &= 200e^{-16}\sqrt{133} \end{aligned}$$

3. Let $g(x,y) = x^2 + y^2 - 4x$.

   (a) Find the gradient vector $\nabla g(1,2)$ and use it to find the tangent line to the level curve $g(x,y) = 1$ at the point $(1,2)$.

   Solution: First, let's find the $\nabla g(1,2)$:

   $$\nabla g(x,y) = \langle 2x-4, 2y\rangle \implies \nabla g(1,2) = \langle -2, 4\rangle.$$

   The tangent line to the level curve at $(1,2)$ is

   $$z = 1 - 2(x-1) + 4(y-2).$$

   (b) Sketch the level curve, the tangent line, and the gradient vector. Label each one clearly.

4. A function is called homogeneous of degree n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second-order partial derivatives.

   (a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

   Solution: We need to show that $f(tx, ty) = t^3 f(x, y)$.

$$\begin{aligned}
f(tx, ty) &= (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 \\
&= t^2x^2ty + 2txt^2y^2 + 5t^3y^3 \\
&= t^3x^2y + 2t^3xy^2 + 5t^3y^3 \\
&= t^3(x^2y + 2xy^2 + 5y^3) \\
&= t^3 f(x, y)
\end{aligned}$$

   Thus, f is homogeneous of degree 3.

   (b) Show that f satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x, y).$$

   Solution: We will start with the left hand side and show that it is equal to the right hand side. Notice that we'll need to find f_x and f_y. First, f_x:

$$f_x(x, y) = 2xy + 2y^2.$$

   Then, f_y:

$$f_y(x, y) = x^2 + 4xy + 15y^2.$$

   Now, on the left hand side we have

$$\begin{aligned}
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x(2xy + 2y^2) + y(x^2 + 4xy + 15y^2) \\
&= 2x^2y + 2xy^2 + x^2y + 4xy^2 + 15y^3 \\
&= 3x^2y + 6xy^2 + 15y^3 \\
&= 3(x^2y + 2xy^2 + 5y^3) \\
&= 3f(x, y)
\end{aligned}$$

   Thus, f satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x, y).$$