Graded by: ____________________ Score: ________ ID: ______________________

WORKSHEET 2 - DUE 9/14

MATH 2110Q – Fall 2015
Professor Hohn

You must show all of your work to receive full credit!

1. Decide whether each of the following expressions make sense. If so, calculate the given expression. If not, explain why.

   Let $\vec{a} = \hat{x} + \hat{y} - 2\hat{z}$, $\vec{b} = 3\hat{x} - 2\hat{y} + \hat{z}$, and $\vec{c} = \hat{y} - 5\hat{z}$.

   (a) $\vec{a} \cdot \vec{b}$

   Solution:
   $$\vec{a} \cdot \vec{b} = 1 \cdot 3 + 1 \cdot (-2) + (-2) \cdot 1 = 3 - 2 - 2 = -1$$

   (b) $\left\|\vec{b} \times \vec{c}\right\|$

   Solution: First, we will find $\vec{b} \times \vec{c}$.
   $$\begin{aligned} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} &= \hat{x}(10 - 1) - \hat{y}(-15 - 0) + \hat{z}(3 - 0) \\ &= \langle 9, 15, 3 \rangle \end{aligned}$$
   Then,
   $$\left\|\vec{b} \times \vec{c}\right\| = \|\langle 9, 15, 3 \rangle\| = \sqrt{9^2 + 15^2 + 3^2} = \sqrt{81 + 225 + 9} = \sqrt{315}$$

   (c) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$

   Solution: Notice that $\vec{b} \cdot \vec{c}$ is a scalar. To solve this expression, we would have to take the dot product of vector a with a scalar, which is impossible. Thus, the expression $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ does not make sense.

   (d) $\vec{a} \times (\vec{b} \times \vec{c})$

   Solution: In part (b), we calculated $\vec{b} \times \vec{c}$. Thus, we need only to calculate the cross product of $\vec{a}$ with that vector.
   $$\begin{aligned} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} &= \hat{x}(3 - (-30)) - \hat{y}(3 - (-18)) + \hat{z}(15 - 9) \\ &= \langle 33, -21, 6 \rangle \end{aligned}$$

2. Find the values of x such that the vectors $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal.

Solution: We know that two vectors are orthogonal when the dot product of the vectors is 0. Let $\vec{a} = \langle 3, 2, x\rangle$ and $\vec{b} = \langle 2x, 4, x\rangle$. We are told that $\vec{a}$ and $\vec{b}$ are orthogonal. Thus,

$$\begin{aligned}
\langle 3, 2, x\rangle \cdot \langle 2x, 4, x\rangle &= 0 \\
3(2x) + 2 \cdot 4 + x \cdot x &= 0 \\
x^2 + 6x + 8 &= 0 \\
(x+4)(x+2) &= 0 \\
\implies x &= -4, -2
\end{aligned}$$

Thus, the values of x such that vectors $\vec{a}$ and $\vec{b}$ are orthogonal are $x = -4, -2$.

3. Let $\vec{a} = \langle 1, 1, -1\rangle$ and $\vec{b} = \langle 2, 4, 6\rangle$

   (a) Compute $\vec{c} = \vec{a} \times \vec{b}$.

   Solution:

$$\begin{aligned}
\vec{c} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} \\
&= \hat{x}(6 - (-4)) - \hat{y}(6 - (-2)) + \hat{z}(4 - 2) \\
&= \langle 10, -8, 2\rangle
\end{aligned}$$

   (b) Show that $\vec{c}$ is orthogonal to $\vec{a}$.

   Solution: We show that $\vec{c}$ is orthogonal to $\vec{a}$ by showing that $\vec{a} \cdot \vec{c} = 0$.

$$\vec{a} \cdot \vec{c} = \langle 1, 1, -1\rangle \cdot \langle 10, -8, 2\rangle = 10 - 8 - 2 = 0$$

   (c) Show that $\vec{c}$ is orthogonal to $\vec{b}$.

   Solution: In a similar manner to part(b),

$$\vec{b} \cdot \vec{c} = \langle 2, 4, 6\rangle \cdot \langle 10, -8, 2\rangle = 20 - 32 + 12 = 0$$

4. Let $\vec{a}, \vec{b} \in V_3$. Show that

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

   Hint: Start by letting $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$. Show that the right hand side of the equation and the left hand side are the same.

Solution: Let $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$. First, let's compute the left hand side of our equation.

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \hat{x}(a_2 b_3 - a_3 b_2) - \hat{y}(a_1 b_3 - a_3 b_1) + \hat{z}(a_1 b_2 - a_2 b_1)
\end{aligned}$$

On the right hand side, we have

$$
\begin{aligned}
-\vec{b} \times \vec{a} &= -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= -\Big(\hat{x}(a_3 b_2 - a_2 b_3) - \hat{y}(a_3 b_1 - a_1 b_3) + \hat{z}(a_2 b_1 - a_1 b_2)\Big) \\
&= \hat{x}(a_2 b_3 - a_3 b_2) - \hat{y}(a_1 b_3 - a_3 b_1) + \hat{z}(a_1 b_2 - a_2 b_1).
\end{aligned}
$$

Thus,

$$\vec{a} \times \vec{b} = \hat{x}(a_2 b_3 - a_3 b_2) - \hat{y}(a_1 b_3 - a_3 b_1) + \hat{z}(a_1 b_2 - a_2 b_1) = -\vec{b} \times \vec{a}.$$