Name: ________________________________

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WORKSHEET 3 - CHAPTER 14 (DUE TUES, MAR 3)

Math 2110Q – Spring 2015
Professor Hohn

You must show all of your work to receive full credit!

1. (a) Find an equation of the tangent plane to the surface $z = xe^{xy}$ at the point $(2, 0, 2)$.

Solution: The equation of a tangent plane to a surface at the point (x_0, y_0) is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(x - x_0) + z_0 = z.$$

First, we find the partial derivative of $f(x, y) = xe^{xy}$ with respect to x.

$$f_x(x, y) = 1 \cdot e^{xy} + x \cdot ye^{xy} = e^{xy} + xye^{xy} = (1 + xy)e^{xy}$$

Note that we needed to use the product rule. Now, we find the partial derivative of $f(x, y) = xe^{xy}$ with respect to y.

$$f_y(x, y) = x \cdot xe^{xy} = x^2e^{xy}$$

At the point $(2, 0, 2)$,

$$f_x(2, 0) = (1 + 2 \cdot 0)e^{2 \cdot 0} = 1, \qquad f_y(2, 0) = 2^2e^{2 \cdot 0} = 4$$

Then, our equation of the plane becomes

$$(x - 2) + 4(y - 0) + 2 = z$$

(b) If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(1, 1)$.

Solution: We rewrite our equation to help us take a derivative: $f(x, y) = (x^3 + y^3)^{1/3}$.

$$f_x(x, y) = (1/3)(x^3 + y^3)^{-2/3}(3x^2) \implies f_x(x, y) = x^2(x^3 + y^3)^{-2/3}$$

So, at the point $(1, 1)$,

$$f_x(1, 1) = 1^2 \cdot (1^3 + 1^3)^{-2/3} = 2^{-2/3}$$

2. If R is the total resistance of there resistors, connected in parallel, with resisitances R_1, R_2,

R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\Omega$, $R_2 = 40\Omega$, $R_3 = 50\Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

Solution: First, let's do a change of variable to clarify what happening. Let $x = R_1, y = R_2, z = R_3, w(x, y, z) = R(R_1, R_2, R_3)$. To find that maximum error, we will look at the differential of w.

$$dw = w_x dx + w_y dy + w_z dz$$

We need to find the partial derivatives of w at the point $(25, 40, 50)$, where

$$\frac{1}{w} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

The partial derivative of w w.r.t. x is

$$\frac{-1}{w^2} \cdot \frac{\partial w}{\partial x} = \frac{-1}{w^2} \implies \frac{\partial w}{\partial x} = \frac{w^2}{x^2} = (w/x)^2.$$

At the point $(25, 40, 50)$,

$$w_x = \left(\frac{\frac{200}{17}}{25}\right)^2 = (8/17)^2 = 64/289.$$

Now, we find the partial derivative of w w.r.t. y.

$$\frac{-1}{w^2} \cdot \frac{\partial w}{\partial y} = \frac{-1}{w^2} \implies \frac{\partial w}{\partial y} = \frac{w^2}{y^2} = (w/y)^2.$$

At the point $(25, 40, 50)$,

$$w_y = \left(\frac{\frac{200}{17}}{40}\right)^2 = (5/17)^2 = 25/289.$$

The partial derivative of w w.r.t. z is

$$\frac{-1}{w^2} \cdot \frac{\partial w}{\partial z} = \frac{-1}{w^2} \implies \frac{\partial w}{\partial z} = \frac{w^2}{z^2} = (w/z)^2.$$

At the point $(25, 40, 50)$,

$$w_z = \left(\frac{\frac{200}{17}}{50}\right)^2 = (4/17)^2 = 16/289.$$

We also need to know dx, dy, dz. Recall that $x = 25$, $y = 40$ and $z = 50$. We also know that the maximum error is 0.5% in x, y, z. Hence, $dx = 25 \cdot 0.005 = 0.125$, $dy = 40 \cdot 0.005 = 0.2$, and $dz = 50 \cdot 0.005 = 0.25$.

Then,

$$dw = (64/289) \cdot 0.125 + (25/289) \cdot 0.2 + (16/289) \cdot 0.25 = 1/17$$

The maximum error in the calculated value for R is then 1/17 ohms.

3. If $z = f(x, y)$, where $x = r\cos\theta$, and $y = r\sin\theta$, find

   (a) $\partial z/\partial r$

   Solution: We need to use the chain rule to find $\partial z/\partial r$.

   $$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}\\ &= \frac{\partial z}{\partial x}\cdot(\cos\theta) + \frac{\partial z}{\partial y}\cdot(\sin\theta)\end{aligned}$$

   (b) $\partial z/\partial\theta$

   Solution: We need to use the chain rule to find $\partial z/\partial\theta$.

   $$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}\\ &= \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta)\end{aligned}$$

4. Show that any function of the form

   $$z = f(x + at) + g(x - at)$$

   is a solution of the wave equation

   $$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

   [Hint: Let $u = x + at$ and $v = x - at$.]

   Solution: First, we rewrite our equation using the hint provided. Let $u = x + at$ and let $v = x - at$. Then, our function becomes $z(u, v) = f(u) + g(v)$. We will start with the left hand side and show that it is equivalent to the right hand side.

   Let's find the partial derivative of z with respect to t.

   $$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}\\ &= \frac{df}{du}\cdot a + \frac{dg}{dv}\cdot(-a)\\ &= a\left(\frac{df}{du} - \frac{dg}{dv}\right)\end{aligned}$$

Now, we find the second partial derivative of z with respect to t.

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} a\left(\frac{df}{du} - \frac{dg}{dv}\right)\\ &= a\left(\frac{d^2 f}{du^2}\frac{\partial u}{\partial t} - \frac{d^2 g}{dv^2}\frac{\partial v}{\partial t}\right)\\ &= a\left(\frac{d^2 f}{du^2}\cdot a - \frac{d^2 g}{dv^2}\cdot(-a)\right)\\ &= a^2\left(\frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\right)\end{aligned}$$

We will show that this is equal to $a^2\dfrac{\partial^2 z}{\partial x^2}$.

The first partial derivative of z with respect to x.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\\ &= \frac{df}{du}\cdot 1 + \frac{dg}{dv}\cdot 1\\ &= \frac{df}{du} + \frac{dg}{dv}\end{aligned}$$

The second partial derivative of z w.r.t. x.

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial t}\left(\frac{df}{du} + \frac{dg}{dv}\right)\\ &= \frac{d^2 f}{du^2}\frac{\partial u}{\partial t} + \frac{d^2 g}{dv^2}\frac{\partial v}{\partial t}\\ &= \frac{d^2 f}{du^2}\cdot 1 - \frac{d^2 g}{dv^2}\cdot 1\\ &= \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\end{aligned}$$

So, indeed

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= a^2\left(\frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\right)\\ &= a^2\frac{\partial^2 z}{\partial x^2}\end{aligned}$$