Name: ______________________

Score: ________ /15

WORKSHEET 5 - CHAPTER 15 (DUE TUES, MAR 31)

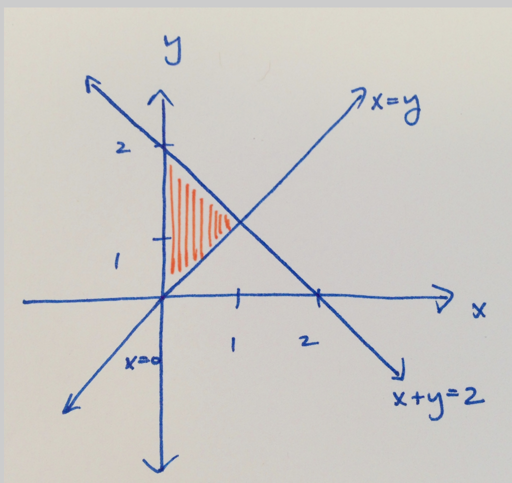
Math 2110Q – Spring 2015
Professor Hohn

You must show all of your work to receive full credit!

1. Find the volume of the solid bounded by the planes $z = x, y = x, x + y = 2$, $x = 0$, and $z = 0$.

Solution: Step 1: Find the region D that we will be using to integrate our function.

The region D we will using to integrate our function $f(x, y) = x$ is the region shown below.



Step 2: Find the order of integration that suits us.

Let's do:

$$\iint_D x\,dA = \int_0^1 \int_x^{2-x} x\,dy\,dx.$$

Step 3: Integrate.

$$\iint_D x\,dA = \int_0^1 \int_x^{2-x} x\,dy\,dx$$
$$= \int_0^1 [xy]_x^{2-x}\,dx$$
$$= \int_0^1 \big[(x(2-x) - x\cdot x\big]\,dx$$
$$= \int_0^1 \big[2x - 2x^2\big]\,dx$$
$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$
$$= 1/3$$

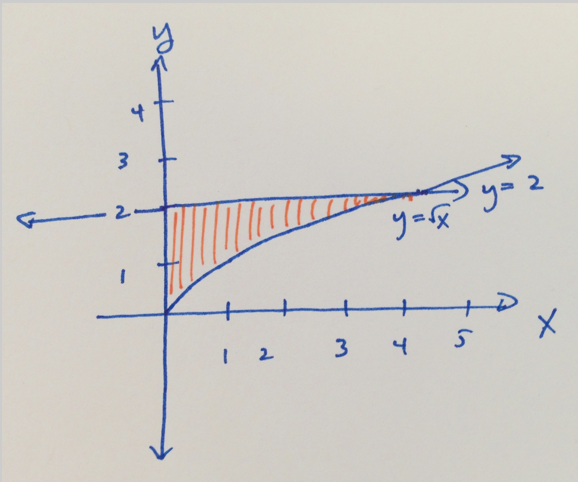
2. Evaluate the integral

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx$$

by reversing the order of integration.

Solution: Step 1: Find the region D that we will were using to integrate our function.

The region D related to the integral is the region shown below.



Step 2: Find the order of integration that suits us.

Let's do:

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy.$$

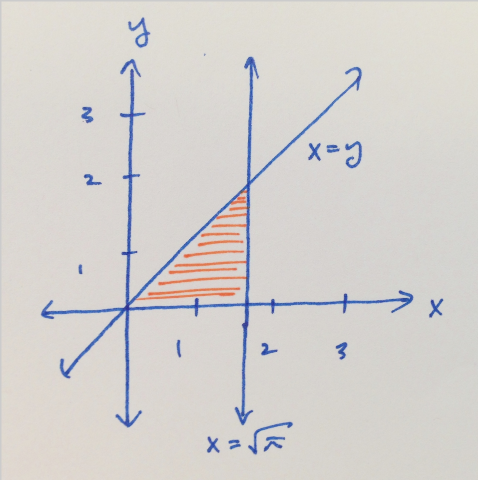
Step 3: Integrate.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$
$$= \int_0^2 \left[\frac{x}{y^3+1}\right]_0^{y^2} dy$$
$$= \int_0^2 \frac{y^2}{y^3+1}\,dy$$
$$= \frac{1}{3}\int_1^9 \frac{1}{u}\,du \qquad \text{where } u = y^3+1$$
$$= \frac{1}{3}\Big[\ln(u)\Big]_1^9$$
$$= \frac{1}{3}\big(\ln 9 - \ln 1\big)$$
$$= \frac{1}{3}\ln 9$$

3. Evaluate the integral
$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy$$
by reversing the order of integration.

Solution: Step 1: Find the region D that we will were using to integrate our function.

The region D related to the integral is the region shown below.



Step 2: Find the order of integration that suits us.

Let's do:
$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy = \int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\,dy\,dx.$$

Step 3: Integrate.

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_0^x \cos(x^2)\, dy\, dx &= \int_0^{\sqrt{\pi}} \left[y\cos(x^2)\right]_0^x\, dx \\
&= \int_0^{\sqrt{\pi}} x\cos(x^2)\, dx \\
&= \frac{1}{2}\int_0^{\pi} \cos u\, du \qquad \text{where } u = x^2,\ du = 2x\, dx \\
&= \frac{1}{2}\Big[\sin u\Big]_0^{\pi} \\
&= \frac{1}{2}\big(\sin\pi - \sin 0\big) \\
&= \frac{1}{2}\cdot 0 \\
&= 0
\end{aligned}$$

4. An alien that is new to our planet (and, hence, new to our crazy calculus rules) is uncertain if the following equations are true. If one is true, explain why. If one is false, explain why or give a counterexample that disproves the statement.

(a)

$$\int_{-1}^{2}\int_0^6 x^2\sin(x-y)\, dx\, dy = \int_0^6\int_{-1}^{2} x^2\sin(x-y)\, dy\, dx$$

Solution: Since $f(x,y) = x^2\sin(x-y)$ is continuous on the rectangle $R = \{(x,y) \mid -1 \le x \le 2, 0 \le y \le 6\}$, by Fubini's Theorem, we can switch the order of integration. Thus,

$$\iint_R f(x,y)\, dA = \int_{-1}^{2}\int_0^6 x^2\sin(x-y)\, dx\, dy = \int_0^6\int_{-1}^{2} x^2\sin(x-y)\, dy\, dx.$$

(b)

$$\int_0^1\int_0^x \sqrt{x+y^2}\, dy\, dx = \int_0^x\int_0^1 \sqrt{x+y^2}\, dx\, dy$$

Solution: These two integrals are not equivalent! The integral on the left hand side is equal to a number, while the integral on the right hand side is equal to a function.

(c)

$$\int_1^2\int_3^4 x^2e^y\, dy\, dx = \int_1^2 x^2\, dx \int_3^4 e^y\, dy$$

Solution: Let $f(x) = x^2$ and let $g(y) = e^y$. Then, we have

$$\int_1^2 \int_3^4 x^2 e^y \, dy \, dx = \int_1^2 \int_3^4 f(x)g(y) \, dy \, dx$$

When we integrate with respect to y, our function $f(x)$ acts like a constant. So, we have

$$\begin{aligned}
\int_1^2 \int_3^4 f(x)g(y) \, dy \, dx &= \int_1^2 f(x) \int_3^4 g(y) \, dy \, dx \\
&= \int_1^2 f(x) \left(\int_3^4 g(y) \, dy \right) dx \\
&= \left(\int_3^4 g(y) \, dy \right) \int_1^2 f(x) \, dx \\
&= \int_1^2 f(x) \, dx \left(\int_3^4 g(y) \, dy \right) \\
&= \int_1^2 x^2 \, dx \int_3^4 e^y \, dy
\end{aligned}$$

Thus, it is true that

$$\int_1^2 \int_3^4 x^2 e^y \, dy \, dx = \int_1^2 x^2 \, dx \int_3^4 e^y \, dy$$

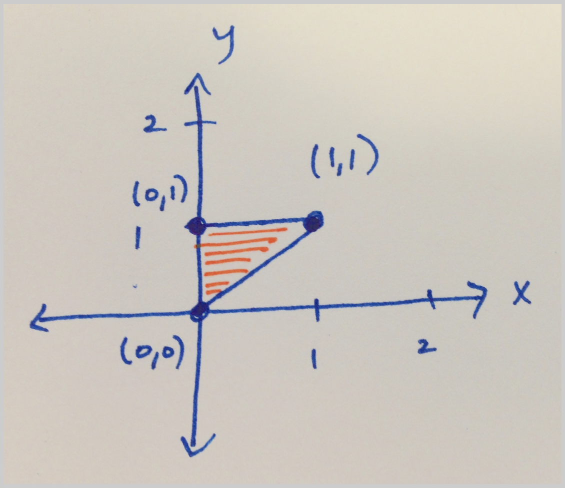
5. Calculate the value of the integral

$$\iint_D \frac{1}{1+x^2} \, dA,$$

where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

Solution: Step 1: Find the region D that we will were using to integrate our function.

The region D related to the integral is the region shown below.



Step 2: Find the order of integration that suits us.

Let's do:

$$\iint_D \frac{1}{1+x^2}\, dA, = \int_0^1 \int_x^1 \frac{1}{1+x^2}\, dy\, dx.$$

Step 3: Integrate.

$$\begin{aligned}
\int_0^1 \int_x^1 \frac{1}{1+x^2}\, dy\, dx &= \int_0^1 \left[\frac{y}{1+x^2}\right]_x^1 dx \\
&= \int_0^1 \frac{1-x}{1+x^2}\, dx \\
&= \int_0^1 \frac{1}{1+x^2}\, dx - \int_0^1 \frac{x}{1+x^2}\, dx \\
&= \int_0^1 \frac{1}{1+x^2}\, dx - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \qquad \text{where } u = 1+x^2,\ du = 2x\, dx \\
&= \int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^2\theta}\, d\theta - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \qquad \text{where } x = \tan\theta,\ dx = \sec^2\theta\, d\theta \\
&= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^2\theta}\, d\theta - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \qquad \text{by trig sub where } 1+\tan^2\theta = \sec^2\theta \\
&= \int_0^{\pi/4} 1\, d\theta - \frac{1}{2}\int_1^2 \frac{1}{u}\, du \\
&= \frac{\pi}{4} - 0 - \frac{1}{2}\Big[\ln u\Big]_1^2 \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$