Name: ______________________

Score: ________ /15

WORKSHEET 6 - CHAPTER 15 (DUE TUES, APR 7)

Math 2110Q – Spring 2015
Professor Hohn

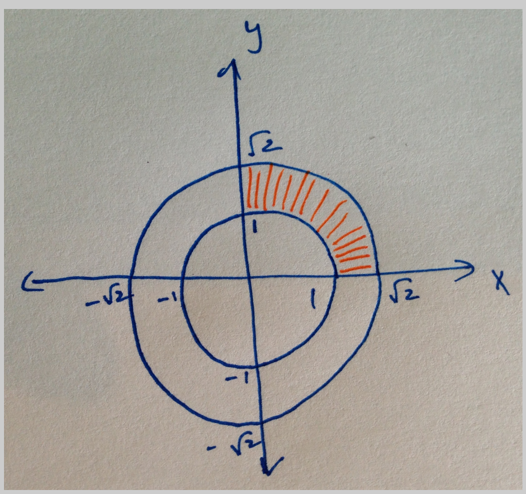
You must show all of your work to receive full credit!

1. Calculate the value of the integral

$$\iint_D x\, dA$$

where D is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.

Solution: Step 1: Draw D
Region D is the region that is between two circles (one of radius 1 and one of radius $\sqrt{2}$) that lies in the first quadrant. Since we have a region that is circular, we will want to use polar coordinates. Then, $1 \le r \le \sqrt{2}$ and $0 \le \theta \le \pi/2$. See the picture below.



Step 2: Set up the integral
From the picture, we see that $1 \le r \le \sqrt{2}$ and $0 \le \theta \le \pi/2$. So, we have

$$\iint_D x\, dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} (r\cos\theta)\, r\, dr\, d\theta.$$

Step 3: Integrate

$$\begin{aligned}
\int_0^{\pi/2}\int_1^{\sqrt{2}} (r\cos\theta)\, r\, dr\, d\theta &= \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\cos\theta\, dr\, d\theta \\
&= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_1^{\sqrt{2}} \cos\theta\, d\theta \\
&= \int_0^{\pi/2} \left(\frac{2^{3/2}}{3} - \frac{1}{3}\right) \cos\theta\, d\theta \\
&= \frac{1}{3}\left(2^{3/2}-1\right)\int_0^{\pi/2} \cos\theta\, d\theta \\
&= \frac{1}{3}\left(2^{3/2}-1\right)\Big[\sin\theta\Big]_0^{\pi/2} \\
&= \frac{1}{3}\left(2^{3/2}-1\right)(1-0) \\
&= \frac{1}{3}\left(2^{3/2}-1\right)
\end{aligned}$$

2. Evaluate the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

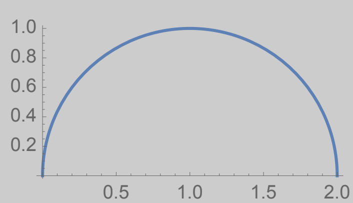
by converting to polar coordinates.

Solution: Step 1: Draw D
First, we use the bounds of our integral to describe the region we will integrate over.

$$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 - 2x + y^2 = 0 \implies (x-1)^2 + y^2 = 1.$$

Our region is then a half circle that is shifted to the right.

The region we will integrate over is



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Step 2: Set up the integral
We set up our integral using polar coordinates. Since $y^2 = 2x - x^2$,

$$x^2 + y^2 = 2x \implies r^2 = 2r\cos\theta \implies r = 2\cos\theta.$$

Therefore, $0 \leq r \leq 2\cos\theta$ and $0 \leq \theta \leq \pi/2$. Recall that $\sqrt{x^2+y^2} = r$. Our integral is

$$\iint_R f(r\cos\theta, r\sin\theta)\cdot r\, dr\, d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r \cdot r\, dr\, d\theta.$$

Step 3: Integrate.

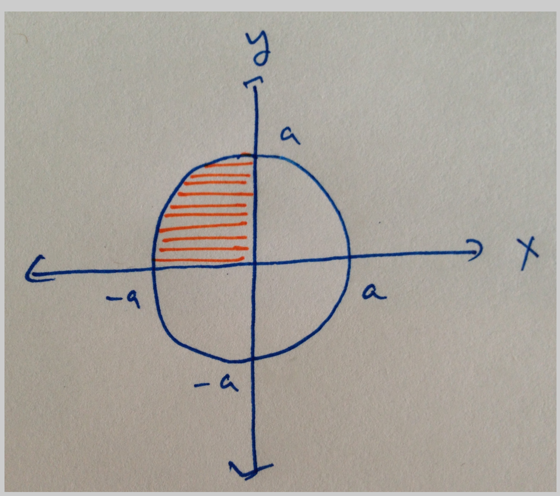
$$\begin{aligned}
\int_0^{\pi/2}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta &= \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta \\
&= \frac{1}{3}\int_0^{\pi/2}(2\cos\theta)^3 d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2}(\cos\theta-\cos\theta\sin^2\theta)\,d\theta \\
&= \frac{8}{3}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_0^{\pi/2} \\
&= \frac{8}{3}\left(\left(1-\frac{1}{3}\right)-(0-0)\right) \\
&= \frac{16}{9}
\end{aligned}$$

3. Evaluate the integral

$$\int_0^a\int_{-\sqrt{a^2-y^2}}^0 x^2y\,dx\,dy$$

by converting to polar coordinates.

Solution: Step 1: Draw D
From the integral given, we know that $-\sqrt{a^2-y^2}\le x\le 0$ and $0\le y\le a$. This means that x has a boundary at $x=0$ and $x=-\sqrt{a^2-y^2}$. The boundary $x=-\sqrt{a^2-y^2}$ is the left hand side of the circle $x^2+y^2=a^2$ (circle of radius a). Since $0\le y\le a$, the region we seek is in the upper half plane. Thus, the region D is a quarter of a circle in the second quadrant with radius a, centered at 0. See picture below.



Step 2: Step up the integral
We will be using polar coordinates. We can see from the picture that $0 \leq r \leq a$ and $\pi/2 \leq \theta \leq \pi$. So, we have

$$\iint_D x^2 y\, dx\, dy = \int_{\pi/2}^{\pi} \int_0^a (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta.$$

Step 3: Integrate

$$\begin{aligned}
\int_{\pi/2}^{\pi} \int_0^a (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta &= \int_{\pi/2}^{\pi} \int_0^a r^4 \cos^2\theta \sin\theta\, dr\, d\theta \\
&= \int_{\pi/2}^{\pi} \left[\frac{r^5}{5}\right]_0^a \cos^2\theta \sin\theta\, d\theta \\
&= \frac{a^5}{5} \int_{\pi/2}^{\pi} \cos^2\theta \sin\theta\, d\theta \\
&= -\frac{a^5}{5} \int_0^{-1} u^2\, du \qquad \text{where } u = \cos\theta,\ du = -\sin\theta\, d\theta \\
&= -\frac{a^5}{5} \left[\frac{u^3}{3}\right]_0^{-1} \\
&= -\frac{a^5}{5} \left(\frac{-1}{3} - \frac{0}{3}\right) \\
&= \frac{a^5}{15}
\end{aligned}$$

4. We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center at the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi.$$

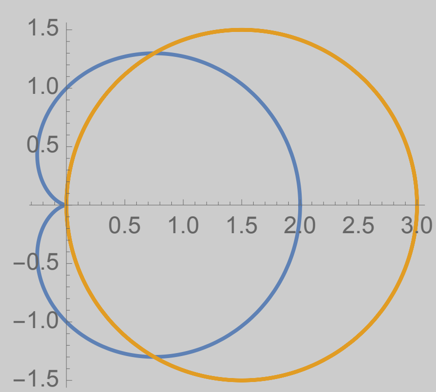
Solution: The region D_a is a circle of radius a. This is the region we are integrating over.

Since we are integrating over a circle, we want to use polar coordinates.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA &= \lim_{a\to\infty}\iint\limits_{D_a} e^{-(x^2+y^2)}\,dA \\
&= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta \\
&= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta \\
&= \lim_{a\to\infty}\int_0^{2\pi}\frac{-1}{2}\left[e^{-r^2}\right]_0^a d\theta \\
&= \lim_{a\to\infty}\int_0^{2\pi}\frac{-1}{2}\left(e^{-a^2}-1\right)d\theta \\
&= \lim_{a\to\infty}\left[\frac{-1}{2}\left(e^{-a^2}-1\right)\theta\right]_0^{2\pi} \\
&= \lim_{a\to\infty}\left(\frac{-1}{2}\left(e^{-a^2}-1\right)2\pi\right) \\
&= \lim_{a\to\infty}-\pi\left(e^{-a^2}-1\right) \\
&= \lim_{a\to\infty}\left(-\pi e^{-a^2}+\pi\right) \\
&= \pi \qquad\qquad \left(\text{because } \lim_{a\to\infty} e^{-a^2}=0\right)
\end{aligned}$$

5. Use a double integral to find the area of the region inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 3\cos\theta$.

Solution: The two functions that describe our region look like



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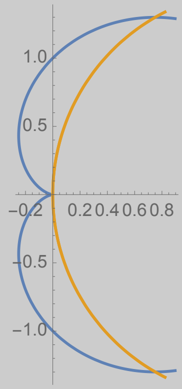
The region we will be integrating over is



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We need to divide our region up into two parts since the equation $r = 3\cos\theta$, θ is bounded by $-\pi/2 \le \theta \le \pi/2$. Moreover, the symmetry along the x-axis allows us to worry about computing the area in the top-half plane and then multiplying that area by 2.

We set up our equation using polar coordinates. Notice that we divided our region in the top-half plane into two pieces.

$$\iint_R dA = 2\left(\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta\right)$$

Now,we solve our integrals. The left one:

$$\begin{aligned}
\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta &= \int_{\pi/3}^{\pi/2}\left[\frac{r^2}{2}\right]_{3\cos\theta}^{1+\cos\theta} d\theta \\
&= \int_{\pi/3}^{\pi/2}\left(\frac{(1+\cos\theta)^2}{2} - \frac{(3\cos\theta)^2}{2}\right) d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{\pi/2}\left(1 + 2\cos\theta - 8\cos^2\theta\right) d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{\pi/2}\left(1 + 2\cos\theta - 4\cos(2\theta) - 4\right) d\theta \qquad \text{because } \frac{\cos(2\theta)+1}{2} = \cos^2\theta \\
&= \frac{1}{2}\int_{\pi/3}^{\pi/2}\left(-3 + 2\cos\theta - 4\cos(2\theta)\right) d\theta \\
&= \frac{1}{2}\Big[-3\theta + 2\sin\theta - 2\sin(2\theta)\Big]_{\pi/3}^{\pi/2} \\
&= \frac{1}{2}\Big((-3\pi/2 + 2(1) - 2(0)) - (-\pi + 2(\sqrt{3}/2) - 2(\sqrt{3}/2))\Big) \\
&= -\pi/4 + 1
\end{aligned}$$

The right one:

$$\begin{aligned}
\int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta &= \int_{\pi/2}^{\pi}\left[\frac{r^2}{2}\right]_{3\cos\theta}^{1+\cos\theta} d\theta \\
&= \int_{\pi/2}^{\pi}\frac{(1+\cos\theta)^2}{2}\,d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)\,d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\cos\theta+\frac{\cos(2\theta)+1}{2}\right)\,d\theta \qquad \text{because } \frac{\cos(2\theta)+1}{2}=\cos^2\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos(2\theta)}{2}\right)\,d\theta \\
&= \frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{\sin(2\theta)}{4}\right]_{\pi/2}^{\pi} \\
&= \frac{1}{2}\left(\left(\frac{3\pi}{2}+2(0)+\frac{0}{4}\right)-\left(\frac{3\pi}{4}+2(1)+\frac{0}{4}\right)\right) \\
&= \frac{1}{2}\Big(\pi/2+2\Big) \\
&= \frac{3\pi}{8}-1
\end{aligned}$$

Then,

$$\iint\limits_R dA = 2\left(\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta+\int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta\right)=\pi/4$$