Name: ______________________

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WORKSHEET 8 - CHAPTER 13, 16 (DUE TUES, APR 28)

Math 2110Q – Spring 2015
Professor Hohn

You must show all of your work to receive full credit!

1. Let $\vec{r}(t) = \langle \sqrt{2-t}, (e^t - 1)/t, \ln(t+1) \rangle$.

   (a) Find the domain of $\vec{r}$. Write your answer using interval notation.

   Solution: Each component of $\vec{r}(t)$ must make sense (be defined). We have then

   $$2 - t \geq 0 \implies t \leq 2,\ t \neq 0, \text{ and } t + 1 > 0 \implies t > -1$$

   Combining these restrictions, we have

   $$t \in (-1, 0) \cup (0, 2]$$

   (b) Find $\lim_{t \to 0} \vec{r}(t)$.

   Solution:

   $$\lim_{t\to 0} \vec{r}(t) = \langle \lim_{t\to 0} \sqrt{2-t}, \lim_{t\to 0} \frac{e^t - 1}{t}, \lim_{t\to 0} \ln(t+1) \rangle = \langle \sqrt{2}, \lim_{t\to 0} \frac{e^t}{1}, \ln(1) \rangle = \langle \sqrt{2}, 1, 0 \rangle$$

   (c) Find $\vec{r}\,'(t)$.

   Solution:

   $$\vec{r}\,'(t) = \langle \frac{1}{2}(2-t)^{-1/2}, \frac{te^t - (e^t - 1)\cdot 1}{t^2}, \frac{1}{t+1} \rangle = \langle \frac{1}{2\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1} \rangle$$

   (d) Find $\vec{T}(t)$ at the point where $t = 1$.

   Solution: At $t = 1$, $\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|}$. Then,

   $$\vec{r}\,'(1) = \langle \frac{1}{2\sqrt{2-1}}, \frac{1e^1 - e^1 + 1}{1^2}, \frac{1}{1+1} \rangle = \langle \frac{1}{2}, 1, \frac{1}{2} \rangle$$

   $$\|\vec{r}\,'(1)\| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}$$

Thus,

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \langle \frac{1}{2\sqrt{\frac{3}{2}}}, \frac{1}{\sqrt{\frac{3}{2}}}, \frac{1}{2\sqrt{\frac{3}{2}}} \rangle = \langle \frac{\sqrt{2}}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{2\sqrt{3}} \rangle$$

2. Find the curvature of the ellipse $x = 3\cos t, y = 4\sin t$ at the points $(3, 0)$ and $(0, 4)$.

Solution: Recall that to find curvature, we can compute it in one of two ways:

$$\kappa(t) = \frac{\left\|\vec{T}\,'(t)\right\|}{|\vec{r}\,'(t)|}$$

or

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

In either case, we need to find $\vec{r}\,'(t)$.

$$\vec{r}\,'(t) = \langle -3\sin t, 4\cos t \rangle$$

Notice that finding $\vec{T}\,'(t)$ may be difficult here. So, we will proceed with the second way to compute curvature. Note that we can rewrite our $\vec{r}(t)$ as

$$\vec{r}(t) = \langle 3\cos t, 4\sin t, 0 \rangle$$

$$\vec{r}\,''(t) = \langle -3\cos t, -4\sin t \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3\sin t & 4\cos t & 0 \\ -3\cos t & -4\sin t & 0 \end{vmatrix} = \langle 0, 0, 12\sin^2 t + 12\cos^2 t \rangle = \langle 0, 0, 12 \rangle$$

Thus,

$$\left\|\vec{r}\,'(t) \times \vec{r}\,''(t)\right\| = \sqrt{0^2 + 0^2 + 12^2} = 12$$

and

$$\left\|\vec{r}\,'(t) = \langle -3\sin t, 4\cos t \rangle\right\|^3 = \left((-3\sin t)^2 + (-4\cos t)^2\right)^{3/2} = \left(9\sin^2 t + 16\cos^2 t\right)^{3/2}$$

Then,

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{12}{\left(9\sin^2 t + 16\cos^2 t\right)^{3/2}}$$

The t values must be found to complete our calculation. Note that we want t such that $x(t) = 3, y(t) = 0$. Thus, $t = 0$ for the point $(3, 0)$. For point $(0, 4)$, $t = \pi/2$. Then,

$$\text{For point } (3,0): \qquad \kappa(0) = \frac{12}{16^{3/2}} = \frac{12}{64} = \frac{3}{16}$$

$$\text{For point } (0,4): \qquad \kappa(\pi/2) = \frac{12}{9^{3/2}} = \frac{12}{27} = \frac{4}{9}$$

3. Find the gradient vector field $\vec{F} = \nabla f$ of $f(x, y) = \sqrt{x^2 + y^2}$, sketch the vector field, and draw two level curves with $k = 1, 2$.

Solution: First, we need to find the gradient of f.

$$\vec{F}(x, y) = \nabla f(x, y) = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \rangle$$

The vector field should look like

4. Evaluate the line integral

$$\int_C x \sin y \, ds$$

where C is the line segment from $(0, 3)$ to $(4, 6)$.

Solution: Let's first find a parameterization of our curve C. Since C is a line segment,

$$\vec{r}(t) = \langle 0, 3 \rangle + t\langle 4 - 0, 6 - 3 \rangle = \langle 4t, 3 + 3t \rangle, \quad 0 \le t \le 1$$

We also need to find $\vec{r}\,'(t)$ and $\|\vec{r}\,'(t)\|$.

$$\vec{r}\,'(t) = \langle 4, 3 \rangle$$

and

$$\left\|\vec{r}\,'(t)\right\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

Then,

$$\begin{aligned} \int_C x \sin y \, ds &= \int_0^1 4t \sin(3 + 3t) \, 5 \, dt \\ &= 20 \int_0^1 t \sin(3t + 3) dt \end{aligned}$$

We need to use integration by parts to deal with $\int t \sin(3t + 3) dt$.

Let $u = t, dv = \sin(3t + t)$ and $du = dt, v = \dfrac{-\cos(3t + 3)}{3}$.

$$\begin{aligned} 20 \int_0^1 t \sin(3t + 3) \, dt &= 20 \left(\left[\frac{-t \cos(3t + 3)}{3} \right]_0^1 + \int_0^1 \frac{\cos(3t + 3)}{3} dt \right) \\ &= \frac{-20 \cos(6)}{3} - 0 + \frac{20}{3} \int_0^1 \cos(3t + 3) \, dt \\ &= \frac{-20 \cos(6)}{3} + \frac{20}{3} \left[\frac{\sin(3t + 3)}{3} \right]_0^1 \\ &= \frac{-20 \cos(6)}{3} + \frac{20}{3} \left(\frac{\sin(6)}{3} - \frac{\sin(3)}{3} \right) \\ &= \frac{-20 \cos(6)}{3} + \frac{20 \sin(6)}{9} - \frac{20 \sin(3)}{9} \end{aligned}$$

5. Evaluate the line integral

$$\int_C e^x \, dx$$

where C its he arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$.

Solution: First, we need to parameterize our curve C. Since $x = y^3$, we can let $y = t$ so that $x = t^3$. Then,

$$\vec{r}(t) = \langle t^3, t \rangle, \quad -1 \leq t \leq 1$$

where t values are found by looking at $(-1, -1)$ and seeing that $y = t = -1$. For the other point $(1, 1)$, $y = t = 1$. We also need dx to solve our integral.

$$dx = 3t^2 \, dt$$

Then,

$$\begin{aligned}
\int_C e^x \, dx &= \int_0^1 e^{t^3} \, 3t^2 \, dt \\
&= 3 \int_0^1 t^2 e^{t^3} \, dt \\
&= \int_0^1 e^u du \quad \text{where } u = t^3, du = 3t^2 dt \\
&= \Big[e^u \Big]_0^1 \\
&= e^1 - e^0 \\
&= e - 1
\end{aligned}$$