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Final Review Worksheet

Math 2110Q – Fall 2014
Professor Hohn

Answers (in no particular order):

$f(x,y)=e^y+xe^{xy}+C$; 2; 3; $\langle -e^{-y}\cos z, -e^{-z}\cos x, -e^{-x}\cos y\rangle$, $-e^{-x}\sin y-e^{-y}\sin z-e^{-z}\sin x$; $-4,-2$; 87; 8; $\frac{2\sqrt{5}}{3}$; $22/\sqrt{26}$; $x+y+z=4$; $\frac{2}{27}(13^{3/2}-8)$; $\langle 4\cos t, 4\sin t, 5-4\cos t\rangle, 0\le t\le 2\pi$; $\frac{2}{9}e^3-\frac{4}{45}$; $1/4$; $\frac{7\pi}{6}$; $1/2\sin 1$; 12π; $\pi/6$; $\frac{64}{9}\pi$; $\frac{1}{12}(5\sqrt{5}-1)$; $\pi/4$; $\frac{1}{2}(3\pi-9)$, $\frac{-3\pi}{4}$; $\langle t^3/3, \frac{1}{\pi}\left(t\sin\pi t+\frac{\cos\pi t}{\pi}\right), \frac{-\cos\pi t}{\pi}\rangle$; $(2,1/2,-1)$, $(-2,-1/2,1)$; 5, 0; $2e^{-1}$, 0; 36, 18, 18; -1, $(0,0)$; $\{(x,y)\mid x^2+y^2\le 4\}$; $\frac{-y}{\sqrt{4-x^2-y^2}}$; $\frac{-x}{\sqrt{4-x^2-y^2}}$; $x+y+\sqrt{2}z=4$; $25/6$; $\sqrt{\frac{593}{16}}$; $\langle\frac{24}{\sqrt{593}},\frac{4}{\sqrt{593}},\frac{1}{\sqrt{593}}\rangle$

1. Find the values of x such that the vectors $\langle 3,2,x\rangle$ and $\langle 2x,4,x\rangle$ are orthogonal.

Solution: For the two vectors to be orthogonal, the dot product of the two vectors must be equal to zero. Thus,

$$\langle 3,2,x\rangle\cdot\langle 2x,4,x\rangle=0 \implies 3\cdot 2x+2\cdot 4+x\cdot x=0 \implies x^2+6x+8=0$$

Then,

$$(x+4)(x+2)=0 \implies x=-4,-2$$

Thus, x is $-4,-2$.

2. A constant force $\vec{F}=3\hat{x}+5\hat{y}+10\hat{z}$ moves an object along the line segment from $(1,0,2)$ to $(5,3,8)$. Find the work done if the distance is measured in meters and force in newtons.

Solution: Since the force here is constant, we can use our old tricks from Chapter 12. Recall

$$W=\vec{F}\cdot\vec{r}$$

where $\vec{r}$ is the displacement vector. In our case, we need to find the displacement vector $\vec{r}$. Let $P = (1, 0, 2)$ and $Q = (5, 3, 8)$. Then,

$$\vec{r} = Q - P = (5 - 1, 3 - 0, 8 - 2) = (4, 3, 6)$$

The work is then

$$W = \langle 3, 5, 10\rangle \cdot \langle 4, 3, 6\rangle = 3 \cdot 4 + 5 \cdot 3 + 10 \cdot 6 = 12 + 15 + 60 = 87\, N \cdot m$$

3. Let $f(x, y) = x^2 + xy$ on $\mathbb{R}^2$, and let $F = \nabla f$. Let C be a curve in $\mathbb{R}^2$ starting at the point $(-1, -1)$ and ending at $(2, 3)$. Find $\displaystyle\int_C F \cdot d\vec{r}$.

Solution: Since $\vec{F}$ is conservative, we can use the Fundamental Theorem of Line Integrals to get

$$\int_C F \cdot d\vec{r} = f(2, 3) - f(-1, -1) = 2^2 + 6 - (1^2 + 1) = 8$$

4. Let $f(x, y) = x^2 + xy$ on $\mathbb{R}^2$ again. If C is the line segment starting at the origin $(0, 0)$ and ending at $(2, -1)$, find the line integral $\int_C f\, ds$.

Solution: Let $P = (0, 0)$ and $Q = (2, -1)$. Then the line segment starting at P and ending at Q can be parameterized by $\vec{r}(t) = t\vec{PQ} + \vec{P} = t\big[(2, -1) - (0, 0)\big] + (0, 0) = (2t, -t)$ for $0 \le t \le 1$.

$$\int_C f\, ds = \int_0^1 f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

We have $f(\vec{r}(t)) = f(2t, -t) = 4t^2 - 2t^2 = 2t^2$ and $\|\vec{r}\,'(t)\| = \|(2, -1)\| = \sqrt{5}$. So,

$$\int_C f\, ds = \int_0^1 f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt = 2\sqrt{5}\int_0^1 t^2\, dt = \frac{2\sqrt{5}}{3}.$$

5. Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

Solution:

$$D = 22/\sqrt{26}$$

6. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.

Solution: $x + y + z = 4$

7. Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.

Solution: Let P_1 be the plane $x + y + z = 1$ and P_2 be the plane $2x - 3y + 4z = 5$. If the planes were parallel, then the normal vectors for each plane would be scalar multiples of each other. If the planes were perpendicular, then the normal vectors of the planes would be perpendicular.

Parallel:

$$\vec{n_1} = \langle 1, 1, 1 \rangle, \vec{n_2} = \langle 2, -3, 4 \rangle$$

Notice that not constant c exists such that $c\langle 1, 1, 1 \rangle = \langle 2, -3, 4 \rangle$. Hence, these planes are not parallel.

Perpendicular:

$$\vec{n_1} \cdot \vec{n_2} = 1 \cdot 2 + 1 \cdot -3 + 1 \cdot 4 = 2 - 3 + 4 = 3 \neq 0$$

Hence, these planes are not perpendicular.

8. Find the length of the curve $\vec{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t) \rangle$, $0 \leq t \leq 1$.

Solution: Recall that the length of a curve can be found by computing

$$L = \int_a^b \|\vec{r}'(t)\| \, dt$$

Since $\vec{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t) \rangle$, $0 \leq t \leq 1$,

$$\vec{r}'(t) = \langle 3t^{1/2}, -2\sin(2t), 2\cos(2t) \rangle, \; 0 \leq t \leq 1$$

and

$$\|\vec{r}'(t)\| = \sqrt{(3t^{1/2})^2 + (-2\sin(2t))^2 + (2\cos(2t))^2} = \sqrt{9t + 4(\cos^2 t + \sin^2 t)} = \sqrt{9t + 4}$$

Then,

$$\begin{aligned} L &= \int_0^1 \|\vec{r}'(t)\| \, dt \\ &= \int_0^1 \sqrt{9t + 4} dt \\ &= \frac{1}{9} \int_4^{13} u^{1/2} du \qquad \text{where } u = 9t + 4, du = 9dt \\ &= \frac{1}{9} \left[\frac{2}{3} u^{3/2} \right]_4^{13} \\ &= \frac{2}{27} \left(13^{3/2} - 8 \right) \end{aligned}$$

The length of the curve is

$$\frac{2}{27} \left(13^{3/2} - 8 \right)$$

9. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

Solution: Notice that since we have a cylinder, we can use the parameterization

$$x = 4\cos t, y = 4\sin t, 0 \leq t \leq 2\pi$$

for the cylinder. This curve will intersect the plane when $x + z = 5$. That is, when $z = 5 - x$.

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 5 - 4\cos t\rangle,\ 0 \leq t \leq 2\pi$$

10. If $\vec{r}(t) = \langle t^2 + t\cos \pi t, \sin \pi t\rangle$, evaluate $\int_0^1 \vec{r}(t)\, dt$.

Solution:

$$\langle t^3/3, \frac{1}{\pi}\left(t\sin \pi t + \frac{\cos \pi t}{\pi}\right), \frac{-\cos \pi t}{\pi}\rangle$$

11. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$.

(a) Find and sketch the domain of the function.

Solution:

$$D_f = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

(b) Sketch 3 level curves of the surface described by the function.

Solution:

(c) Find the first partial derivatives of the function.

Solution:

$$f_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}$$

$$f_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}$$

(d) Find the tangent plane of the surface described by the function at the point $(1, 1, \sqrt{2})$

Solution:

$$x + y + \sqrt{2}z = 4$$

12. Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

Solution:

$$(2, 1/2, -1), (-2, -1/2, 1)$$

13. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\partial v/\partial s$ and $\partial v/\partial t$ when $s = 0$ and $t = 1$.

Solution:

$$5, 0$$

14. Let $f(x, y, z) = x^2 y + x\sqrt{1 + z}$.

(a) Find the directional derivative of f at the point $(1, 2, 3)$ in the direction $v = 2\hat{x} + \hat{y} - 2\hat{z}$.

Solution:

$$25/6$$

(b) Find the maximum rate of change of f at the point $(1, 2, 3)$.

Solution:

$$\sqrt{\frac{593}{16}}$$

(c) In what direction does the maximum rate of change occur? Write you answer as a unit vector.

Solution:

$$\langle \frac{24}{\sqrt{593}}, \frac{4}{\sqrt{593}}, \frac{1}{\sqrt{593}} \rangle$$

15. Find the absolute maximum and minimum values of $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ on the disk $x^2 + y^2 \leq 4$.

Solution:

$$2e^{-1}, 0$$

16. A package in the shape of a rectangular box can be mailed by USPS if the sum of its length and girth (the perimeter of a cross section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

Solution:

$$36, 18, 18$$

17. Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^3 - 6xy + 8y^3$.

Solution:

$$-1, (0, 0)$$

18. Calculate the integral:

(a) $\int_0^1 \int_x^{e^x} 3xy^2 \, dy \, dx$

Solution:

$$\frac{2}{9}e^3 - \frac{4}{45}$$

(b) $\int_0^1 \int_0^y \int_x^1 6xyz \, dz \, dx \, dy$

Solution:

$$1/4$$

19. Describe/sketch the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

and evaluate the integral.

Solution:

$$\frac{7\pi}{6}$$

20. Calculate the iterated integral by first reversing the order of integration.

$$\int_0^1 \int_x^1 \cos(y^2) \, dy \, dx$$

Solution:

$$1/2 \sin 1$$

21. Find the volume of the solid that is bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

Solution:

$$12\pi$$

22. Find the volume of the solid that is above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.

Solution:

$$\pi/6$$

23. Convert the following integral into an integral with spherical coordinates.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

Solution:

$$\frac{64}{9}\pi$$

24. Evaluate the line integral

$$\int_C x\,ds$$

where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution:

$$\frac{1}{12}(5\sqrt{5}-1)$$

25. Compute the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

where $F(x,y) = \langle xy, x^2\rangle$ and C is given by $\vec{r}(t) = \langle \sin t, 1+t\rangle$, $0 \le t \le \pi$.

Solution:

$$\pi/4$$

26. Find the work done by the force field

$$\vec{F}(x,y,z) = z\hat{x} + x\hat{y} + y\hat{z}$$

in moving a particle from the point $(3,0,0)$ to the point $(0,\pi/2,3)$ along

(a) a straight line

Solution:

$$\frac{1}{2}(3\pi-9)$$

(b) the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$.

Solution:

$$\frac{-3\pi}{4}$$

27. Show that $F(x,y) = \langle (1+xy)e^{xy}, e^y + x^2e^{xy}\rangle$ is a conservative vector field. Then, find a function f such that $\vec{F} = \nabla f$.

Solution: First we will show that $\vec{F}$ is conservative by showing that if $\vec{F} = \langle P, Q \rangle$, $P_y = Q_x$.

$$P_y = (1 + xy)xe^{xy} + xe^{xy} = 2xe^{xy} + x^2ye^{xy}, \quad Q_x = x^2ye^{xy} + 2xe^{xy}$$

So, $P_y = Q_x$ and $\vec{F}$ is conservative. Now, we will find our potential function f. Here, I will use the comparison method.

$$\int P\,dx = \int (e^{xy} + xye^{xy})\,dx = \frac{e^{xy}}{y} + xe^{xy} - \frac{e^{xy}}{y} + g(y) = xe^{xy} + g(y)$$

where we had to integrate by parts to integrate the xye^{xy} part.

$$\int Q\,dy = \int \left(e^y + x^2e^{xy}\right)dy = e^y + xe^{xy} + h(x)$$

Our function f must be

$$f(x, y) = e^y + xe^{xy} + C$$

28. If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl$\vec{F} = 0$, then $\vec{F}$ is a conservative vector field. Use this fact to show that $\vec{F}(x, y, z) = \langle e^y, xe^y + e^z, ye^z \rangle$ is conservative. Use the fact that $\vec{F}$ is conservative to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve C where C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$.

Solution:

2

29. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0, 0), (1, 0)$, and $(1, 3)$.

Solution:

3

30. Let $\vec{F}(x, y, z) = \langle e^{-x} \sin y, e^{-y} \sin z, e^{-z} \sin x \rangle$.

(a) Find the curl$\vec{F}$.

Solution:

$$\nabla \times \vec{F} = \langle -e^{-y} \cos z, -e^{-z} \cos x, -e^{-x} \cos y \rangle$$

(b) Find the div$\vec{F}$.

Solution:

$$\nabla \cdot \vec{F} = -e^{-x} \sin y - e^{-y} \sin z - e^{-z} \sin x$$