Elementary Differential Equations

Math 2410Q (211Q)– Fall 2013

Professor Hohn

Section 6.3:

28. Consider the differential equation

$$\frac{dy^2}{dt^2} - y = e^{2t}, \quad y(0) = 1, \quad y'(0) = -1\,.$$

(a) Compute the Laplace transform of both sides of the differential equation

Solution: First, apply the Laplace transform to both sides of the equation.

$$\mathcal{L}\left[\frac{dy^2}{dt^2} - y\right] = \mathcal{L}\left[e^{2t}\right]$$

$$\mathcal{L}\left[\frac{dy^2}{dt^2}\right] - \mathcal{L}\left[y\right] = \frac{1}{s-2}\,.$$

Now, we need to compute the Laplace transform of $\mathcal{L}\left[\frac{dy^2}{dt^2}\right]$.

$$\begin{aligned}
\mathcal{L}\left[\frac{dy^2}{dt^2}\right] &= \int_0^\infty \frac{d^2y}{dt^2} e^{-st} dt \\
&= \lim_{b\to\infty} \int_0^b y'' e^{-st} dt \\
&= \lim_{b\to\infty} \left\{ y' e^{-st}\Big|_{t=0}^{t=b} + s\int_0^b y' e^{-st} dt \right\} \quad \text{(integration by parts)} \\
&= -y'(0) + s\mathcal{L}\left[y'\right] \\
&= -y'(0) + s\left(s\mathcal{L}\left[y\right] - y(0)\right) \\
&= s^2\mathcal{L}\left[y\right] - sy(0) - y'(0)
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathcal{L}\left[\frac{dy^2}{dt^2}\right] - \mathcal{L}\left[y\right] &= \frac{1}{s-2} \\
\left(s^2\mathcal{L}\left[y\right] - sy(0) - y'(0)\right) - \mathcal{L}\left[y\right] &= \frac{1}{s-2} \\
\mathcal{L}\left[y\right]\left(s^2-1\right) - sy(0) - y'(0) &= \frac{1}{s-2}
\end{aligned}$$

(b) Substitute in the initial conditions and simplify to obtain the Laplace transform of the solution

Solution:

$$\begin{aligned}
\mathcal{L}[y](s^2-1) - sy(0) - y'(0) &= \frac{1}{s-2} \\
\mathcal{L}[y](s^2-1) - s + 1 &= \frac{1}{s-2} \\
\mathcal{L}[y](s^2-1) &= \frac{1}{s-2} + s - 1 \\
\mathcal{L}[y] &= \frac{1}{(s-2)(s^2-1)} + \frac{s-1}{s^2-1} \\
\mathcal{L}[y] &= \frac{1}{(s-2)(s-1)(s+1)} + \frac{1}{s+1}
\end{aligned}$$

We simplify further by using partial fractions.

$$\begin{aligned}
\frac{1}{(s-2)(s-1)(s+1)} &= \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s-1} \\
&= \frac{A(s+1)(s-1)}{s-2} + \frac{B(s-2)(s-1)}{s+1} + \frac{C(s-2)(s+1)}{s-1} \\
&= \frac{As^2 - A}{s-2} + \frac{B(s^2-3s+2)}{s+1} + \frac{C(s^2-s-2)}{s-1}
\end{aligned}$$

We have to solve the following system:

$$\begin{aligned}
A + B + C &= 0 \\
-3B - C &= 0 \\
-A + 2B - 2C &= 1
\end{aligned}$$

So, we should find that $A = 1/3$, $B = 1/6$, and $C = -1/2$. Therefore, we have

$$\begin{aligned}
\mathcal{L}[y] &= \frac{1}{(s-2)(s-1)(s+1)} + \frac{1}{s+1} \\
&= \frac{\frac{1}{3}}{s-2} + \frac{\frac{1}{6}}{s+1} + \frac{-\frac{1}{2}}{s-1} + \frac{1}{s+1} \\
&= \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{7}{6}\left(\frac{1}{s+1}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right)
\end{aligned}$$

(c) Find the solution by taking the inverse Laplace transform.

Solution: We take the inverse Laplace transform of both sides of our equation from

Part (b).

$$\mathcal{L}^{-1}\left[\mathcal{L}\left[y\right]\right] = \mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{7}{6}\left(\frac{1}{s+1}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right)\right]$$
$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{7}{6}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$
$$y(t) = \frac{1}{3}e^{2t} + \frac{7}{6}e^{-t} - \frac{1}{2}e^{t}$$