Name: ______________________

Midterm 2

Math 2710 – Spring 2014

Professor Hohn

Instructions: Turn off and put away your cell phone. No calculators or electronic devices are allowed. Show all of your work! No credit will be given for unsupported answers or illegible solutions.

Page	1	2	3	4	5	Total
Definitions	8	8	8	8	8	40
Score:						

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Proofs	12	12	12	12	12	60
Score:						

Total score:

1. (8 points) Define what it means for two functions to be equal.

Solution: A function $f : A \to B$ is equal to a function $g : C \to D$ if $A = C$, $B = D$, and $f(x) = g(x)$ for all $x \in A$.

2. (12 points) Use induction to prove Bernoulli's inequality: If $1 + x > 0$, then

$$(1 + x)^n \geqslant 1 + nx$$

for all $n \in \mathbb{N}$.

Solution: (via induction on n) Suppose $1 + x > 0$ and let $P(n)$ be the statement

$$(1 + x)^n \geqslant 1 + nx.$$

We will show $P(n)$ is true for all $n \in \mathbb{N}$. First, we show that the base case is true.

Let $n = 1$. Then, $(1 + x)^1 = 1 + x = 1 + 1 \cdot x$. Therefore, $P(1)$ is true. Now, we will show that if $P(k)$ is true for some $k \in \mathbb{N}$, then $P(k + 1)$ is true.

Assume $P(k)$ is true. Then, $(1 + x)^k \geqslant 1 + kx$. We want to show that $P(k + 1)$ is true.

$$\begin{aligned}
(1 + x)^{k+1} &= (1 + x) \cdot (1 + x)^k \\
&\geqslant (1 + x) \cdot (1 + kx) \quad \text{by } P(k) \\
&= 1 + kx + x + kx^2 \\
&= 1 + (k + 1)x + kx^2 \\
&\geqslant 1 + (k + 1)x \quad \text{since } kx^2 \geqslant 0 \text{ for all } k \in \mathbb{N} \text{ and } x \in \mathbb{R}.
\end{aligned}$$

Therefore, $P(k+1)$ is true. Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

3. (8 points) What is the definition of an equivalence relation?

Solution: A relation R on a set S is an equivalence relation if R is reflexive (for all $x \in S$, xRx), symmetric (if xRy, then yRx for $x, y \in S$), and transitive (if xRy and yRz, then xRz for $x, y, z \in S$).

4. (12 points) Let S be a set and $\mathcal{P}(S)$ the power set of S. For sets $A, B \subseteq \mathcal{P}(S)$, we say that $A \sim B$ if there exists a bijective function $f : A \to B$. Show that $\sim$ is an equivalence relation.

Solution: The relation $\sim$ is an equivalence relation if the relation is reflexive, symmetric, and transitive. First, we will show that the relation $\sim$ is reflexive. That is, we will show that for all $A \subseteq \mathcal{P}(S)$, $A \sim A$.

Let $A \subseteq \mathcal{P}(S)$, and let $i : A \to A$ be the identity mapping that takes any element of A to itself. This mapping is bijective, and hence, $A \sim A$. Now, we will show that $\sim$ is symmetric.

Let $A, B \subseteq \mathcal{P}(S)$, and suppose $A \sim B$. Then, there exists a function $f : A \to B$ that is bijective. Since f is bijective, the inverse of f, $f^{-1} : B \to A$, exists and is bijective. Thus, $B \sim A$. Now, we will show that $\sim$ is transitive.

Let $A, B, C \subseteq \mathcal{P}(S)$, and suppose $A \sim B$ and $B \sim C$. Then, there exist bijections $f : A \to B$ and $g : B \to C$. Since both f and g are bijective, the composition of the functions g and f, call it $h = g \circ f$, is bijective. Hence, $h : A \to C$ is a bijective function and therefore and $A \sim C$.

Since the relation $\sim$ is reflexive, symmetric, and transitive, the relation $\sim$ is an equivalence relation.

5. (8 points) Explain what it means for two sets have the same cardinality.

Solution: Two sets A and B have the same cardinality, written $\#A = \#B$, if there exists a bijective function f from A to B.

6. (12 points) Let S and T be finite sets. Prove that if $\#(T - S) = \#(S - T)$, then $\#S = \#T$.

Solution: Let S and T be finite sets, and suppose $\#(T - S) = \#(S - T)$. We can rewrite the set S as

$$S = (S - T) \cup (S \cap T).$$

Similarly, we can rewrite the set T as

$$T = (T - S) \cup (S \cap T).$$

Since S and T are finite, the sets $(S - T) \subseteq S$, $(T - S) \subseteq T$, and $S \cap T \subseteq S$ are finite. Furthermore, the sets $S - T$ and $S \cap T$ are disjoint, and the sets $T - S$ and $S \cap T$ are disjoint. Hence,

$$\#S = \#(S - T) + \#(S \cap T),$$

and

$$\#T = \#(T - S) + \#(S \cap T).$$

Therefore,

$$\begin{aligned} \#S &= \#(S - T) + \#(S \cap T) \\ &= \#(T - S) + \#(S \cap T) \qquad \text{(by the assumption that } \#(T - S) = \#(S - T)\text{)} \\ &= \#T \end{aligned}$$

Thus, if S and T are finite sets and $\#(T - S) = \#(S - T)$, then $\#S = \#T$.

7. (8 points) Define what it means for a function f to be injective.

Solution: Let A and B be sets. A function $f : A \to B$ is injective if for $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

8. (12 points) Suppose that $f : A \to B$ is any function. Then, a function $g : B \to A$ is called a *left inverse* for f if $g(f(x)) = x$ for all $x \in A$. Prove that f has a left inverse iff f is injective.

Solution: ($\implies$): Suppose a function $f : A \to B$ has a left inverse. Then, there exists a function $g : B \to A$ such that $g(f(x)) = x$ for all $x \in A$. We want to show that f is injective.

Let $a_1, a_2 \in A$ and suppose $f(a_1) = f(a_2)$. If we compose both sides of our equation with g - the left inverse of f, we have $g(f(a_1)) = g(f(a_2))$. Since g is a left inverse, $g(f(a_1)) = a_1$ and $g(f(a_2)) = a_2$. Hence, $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Thus, since $f(a_1) = f(a_2)$ implies $a_1 = a_2$, f is injective.

($\impliedby$): Suppose a function $f : A \to B$ is injective. We want to define a function $g : B \to A$ that acts as a left inverse of f. Let $C = f(A)$ and fix some $a_0 \in A$.

Let $g : B \to A$ be defined in the following manner:

$$g(b) = \begin{cases} a & \text{if } b \in C \text{ and } f(a) = b \\ a_0 & \text{otherwise} \end{cases}$$

First, we must verify that g is a function. Let $b \in B$.

Suppose first that $b \in C$. Since $C = f(A)$, then there is at least one element $a \in A$ such that $f(a) = b$. Suppose that $a_1, a_2 \in A$ such that $g(b) = a_1$ and $g(b) = a_2$. By the definition of g, this means that $f(a_1) = b$ and $f(a_2) = b$; since f is injective, this implies that $a_1 = a_2$, and therefore for each $b \in C$, there exists exactly one element $a \in A$ such that $g(b) = a$.

Suppose now that $b \notin C$. Then $g(b) = a_0$. Therefore, for every $b \notin C$, there exists exactly one element $a_0 \in A$ such that $g(b) = a_0$.

Since $B = C \cup C^c$, this shows that for every $b \in B$, there exists exactly one element $a \in A$ such that $g(b) = a$, meaning that g is a function.

It remains to show that g is the left inverse of f. Let $a \in A$ and $b = f(a)$. Then $g(b) = a$ by definition. Since $g(b) = g(f(a))$, we see that $g(f(a)) = a$, and therefore g is the left inverse of f.

9. (8 points) Define what it means for a function f to be surjective.

Solution: Let A and B be sets. A function $f : A \to B$ is surjective if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$.

10. (12 points) Let A and B be finite sets. Prove that there exists a surjection $f : A \to B$ iff $\#A \geqslant \#B$.

Solution: ($\Longrightarrow$) : Let A and B be finite sets. Let $\#A = n$ and $\#B = m$ for $n, m \in \mathbb{N}$. Suppose there exists a surjection $f : A \to B$. We will show via contradiction that $\#A \geqslant \#B$.

Suppose that there exists a surjection $f : A \to B$. Let us label the elements of A by $A = \{a_1, a_2, ..., a_n\}$. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_n)\} = B$ (where $f(A) = B$ because f is presumed to be a surjection). Let $b_1 = f(a_1), b_2 = f(a_2), ...,$ and $b_n = f(a_n)$ (note that since we have not assumed that f is injective, it could happen that $b_i = b_j$ even if $i \neq j$). We now have $f(A) = \{b_1, b_2, ..., b_n\} = B$. Therefore, since the number of elements in $\{b_1, b_2, ..., b_n\}$ is less than or equal to n (less than in the case $b_i = b_j$ even when $i \neq j$), and $\#\{b_1, b_2, ..., b_n\} = \#B = m$, we have that $m \leqslant n$.

($\Longleftarrow$): As before, let A and B be finite sets, and let $\#A = n$ and $\#B = m$ for $n, m \in \mathbb{N}$. Assume $\#A \geqslant \#B$. Label the elements of A as $A = \{a_1, a_2, ..., a_n\}$; label the elements of B as $B = \{b_1, b_2, ..., b_m\}$. We will now construct a surjection $f : A \to B$.

Define $f : A \to B$ as follows:

$$f(a_i) = \begin{cases} b_i & \text{if } 1 \leqslant i \leqslant m \\ b_1 & \text{if } m + 1 \leqslant i \leqslant n. \end{cases}$$

It remains to show that f is a surjection. Let $b \in B$. Then, there exists some $i_0 \in \mathbb{N}$ in the range $1 \leqslant i_0 \leqslant m$ such that $b = b_{i_0}$. Since $m \leqslant n$, this implies that i_0 is in the range $1 \leqslant i_0 \leqslant n$. Therefore a_{i_0} exists in A and $f(a_{i_0}) = b_{i_0} = b$. Since b was chosen arbitrarily, this proves that for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$, i.e., f is surjective. Thus, if $\#A \geqslant \#B$, there exists a surjection from A to B.