Final - Preview

Math 2710 – Spring 2014
Professor Hohn

Instructions: These exercises are to be worked on alone! You may use your notes and your textbooks for the course, but you are **not allowed to ask for, receive, nor give others assistance on these exercises** which includes asking the internet, Fields medalists, and/or other textbooks.

1. Let R and S be relations on a set A. Prove or give a counterexample for the following:
   (a) If R and S are equivalence relations, then $R \cap S$ is an equivalence relation.
   (b) If R and S are equivalence relations, then $R \cup S$ is an equivalence relation.
2. Using the definition of a convergent sequence, prove the following:
   (a) For any real number k, $\lim_{n\to\infty} \frac{k}{n} = 0$.
   (b) For any real number $k > 0$, $\lim_{n\to\infty} \frac{1}{n^k} = 0$.
3. Fix $n > 0$. Integers a and b are said to be *congruent modulo* n if $a - b$ is divisible by n. That is,
$$a \equiv b \pmod{n} \quad \text{means} \quad n \mid (a-b)\,.$$
   (a) Prove that congruence modulo n is an equivalence relation on $\mathbb{Z}$.
   (b) Given any integer $a \in \mathbb{Z}$ let $[a]$ denote the equivalence class of a under the modulo n equivalence proved in the previous part. The division algorithm tells us that there is some $q, r \in \mathbb{Z}$ such that $a = qn + r$ and $0 \leqslant r \leqslant n - 1$. Show that $[a] = [r]$.
4. Prove that $a^2 \mid b^2$ if and only if $a \mid b$.
5. Let A and B be finite sets. Prove that $\#(A - B) = \#A - \#B$ if and only if $B \subseteq A$.
6. Use induction to prove that for every integer $n \geqslant 2$
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}.$$
7. Suppose $f : A \to B$. Define a relation R on A by xRy if and only if $f(x) = f(y)$.
   (a) Prove that R is an equivalence relation.
   (b) For any $x \in A$, let E_x be the equivalence class of x. That is,
$$E_x = \{y \in A \mid yRx\}.$$

Let E be the collection of all equivalence classes. That is,

$$E = \{E_x \mid x \in A\}.$$

Prove that the function $g : A \to E$ defined by $g(x) = E_x$ is surjective.

(c) Prove that the function $h : E \to B$ defined by $h(E_x) = f(x)$ is injective.

(d) Prove that $f = h \circ g$. That is, $f(x) = h(g(x))$ for all $x \in A$. Thus, we can conclude that any function can be written as the composition of a surjective function and an injective function.

8. Let G be a group, and let $f : G \to G$ be defined by $f(a) = a^{-1}$.
   (a) Show that f is a bijection from G to G.
   (b) Find f^{-1}. (It might help to prove that $(a^{-1})^{-1} = a$.)

9. Let P and Q be propositions. Bucky, Mia, and Wolverine are trying to show that $P \implies Q$.
   (a) Mia shows that $\sim Q \implies \sim P$. Is she done? Why or why not? You may use truth tables to support your answer.
   (b) Bucky shows that $\sim P \implies \sim Q$. Is he done? Why or why not? You may use truth tables to support your answer.
   (c) Wolverine shows that $P \wedge \sim Q \implies False$. Is he done? Why or why not? You may use truth tables to support your answer.

10. Find the limit of the sequence $\{s_n\}$ given by $s_n = \dfrac{(-1)^n}{n+3}$, and prove that the sequence converges to that limit.

11. Let A and B be sets. The *symmetric difference* of A and B is denoted $A \Delta B$ and is defined by

$$A \Delta B = (A - B) \cup (B - A).$$

   (a) Prove that $A \Delta B \subseteq A$ iff $B \subseteq A$.
   (b) Prove that $A \Delta B \subseteq B$ iff $A \subseteq B$.
   (c) Prove that if A and B are finite sets, then $\#(A \Delta B) \leqslant \#A + \#B$ with equality iff $A \cap B = \varnothing$.
   (d) Show by counterexample that the following proposition is false: Given any finite sets A and B, either $\#(A \Delta B) \leqslant \#A$ or $\#(A \Delta B) \leqslant \#B$.

12. Let G be a group under the binary operation $\star$ and let $a_1, a_2, \ldots, a_n$ be elements in $(G, \star)$. Use induction to show that for all $n \geqslant 2$,

$$(a_1 \star a_2 \star a_3 \star \ldots \star a_n)^{-1} = a_n^{-1} \star a_{n-1}^{-1} \star \ldots \star a_2^{-1} \star a_1^{-1}.$$

13. Math Joke: A mathematician runs into the hospital delivery room right as his wife delivers their first baby. His exhausted wife looks up at him and asks, "Is it a boy or a girl?" After a long pause, the mathematician answers, "Yes." Explain this (fantastic) joke.