Midterm 2 - Preview

Math 2710 – Spring 2014

Professor Hohn

Instructions: These exercises are to be worked on alone! You may use your notes and your textbooks for the course, but you are **not allowed to ask for, receive, nor give others assistance on these exercises** which includes asking the internet, Fields medalists, and/or other textbooks.

1. Use induction to prove Bernoulli's inequality: If $1 + x > 0$, then

$$(1 + x)^n \geqslant 1 + nx$$

   for all $n \in \mathbb{N}$.

2. Let S and T be finite sets. Prove that if $\#(T - S) = \#(S - T)$, then $\#S = \#T$.

3. Define a relation R on the set of all integers $\mathbb{Z}$ by xRy iff $x - y = 2k$ for some integer k. Verify that R is an equivalence relation and describe the equivalence class $E_5 = \{s \in \mathbb{Z} : sR5\}$.

4. Suppose that $f : A \to B$. Let $C \subseteq A$ and $D \subseteq B$. Recall that

$$f^{-1}(D) = \{x \in A \mid f(x) \in D\} \subseteq A.$$

   Show that the following hold:
   (a) $C \subseteq f^{-1}[f(C)]$,
   (b) $f[f^{-1}(D)] \subseteq D$.

5. Suppose that $f : A \to B$ and suppose that $C \subseteq A$ and $D \subseteq B$.
   (a) Prove or give a counterexample: $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$.
   (b) If f is injective, then $f^{-1}[f(C)] = C$.
   (c) If f is surjective, then $f[f^{-1}(D)] = D$.

6. Let A and B be finite sets.
   (a) Prove that there exists a surjection $f : A \to B$ iff $\#A \geqslant \#B$.
   (b) Prove that every subset of a finite set is finite.

7. Suppose that $f : A \to B$ is any function. Then, a function $g : B \to A$ is called a *left inverse* for f if $g(f(x)) = x$ for all $x \in A$. Similarly, a function $h : B \to A$ is called a *right inverse* for f if $f(h(y)) = y$ for all $y \in B$.
   (a) Prove that f has a left inverse iff f is injective.
   (b) Prove that f has a right inverse iff f is surjective.

8. Let $f_1 : A_1 \to A_2$, $f_2 : A_2 \to A_3$, …, $f_n : A_n \to A_{n+1}$ be bijective functions. Use induction to show that the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$ that maps A_1 to A_{n+1} is a bijective function and

$$(f_n \circ f_{n-1} \circ \cdots \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}.$$

9. Let S be a set and $\mathcal{P}(S)$ the power set of S. For sets $A, B \subseteq \mathcal{P}(S)$, we say that $A \sim B$ if there exists a bijective function $f : A \to B$. Show that $\sim$ is an equivalence relation.

10. Let A, B, and C be finite sets. Suppose that $A \subseteq B \subseteq C$ and $\#A = \#C$. Prove that $\#A = \#B$ and $\#B = \#C$.

11. Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

12. Let A and B be sets. The *symmetric difference* of A and B is denoted $A\Delta B$ and is defined by

$$A\Delta B = (A - B) \cup (B - A).$$

   (a) Prove that $A\Delta B \subseteq A$ iff $B \subseteq A$.

   (b) Prove that $A\Delta B \subseteq B$ iff $A \subseteq B$.

   (c) Prove that if A and B are finite sets, then $\#(A\Delta B) \leqslant \#A + \#B$ with equality iff $A \cap B = \varnothing$.

   (d) Show by counterexample that the following proposition is false: Given any finite sets A and B, either $\#(A\Delta B) \leqslant \#A$ or $\#(A\Delta B) \leqslant \#B$.

Definition Clarification (Practice)

Determine if each statement is true or false. Justify your answer.

1. Sets:
   (a) If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B.
   (b) The empty set is a subset of every set.
   (c) If $A \cap B = \varnothing$, then either $A = \varnothing$ or $B = \varnothing$.
   (d) If $a \in A - B$, then $x \in A$ or $x \notin B$.

2. Relations:
   (a) $(a, b) = (c, d)$ iff $a = c$ and $b = d$.
   (b) A relation between A and B is an order set subset of $A \times B$.
   (c) In any relation R on a set S, we always have xRx for all $x \in S$.
   (d) $A \times B = \Big\{\{a, b\} : a \in A \text{ and } b \in B\Big\}$.
   (e) A relations is an equivalence relation if it is reflexive, symmetric, and transitive.
   (f) If R is a relation on S, then $\{y \in S : yRx\}$ determines a partition of S.

3. Induction:
   (a) If S is nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \geqslant k$ for all $k \in S$.
   (b) A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction hypothesis.
   (c) Suppose m is a natural number greater than 1. To prove $P(k)$ is true for all $k \geqslant m$, we must first show that $P(k)$ is false for all k such that $1 \leqslant k < m$.

4. Functions:
   (a) A function from A into B is a nonempty relation $f \subseteq A \times B$ such that if $(a, b) \in f$ and $(a, \hat{b}) \in f$ then $b = \hat{b}$.
   (b) A function $f : A \to B$ is injective if for all a and $\hat{a}$ in A, $f(a) = f(\hat{a})$ implies that $a = \hat{a}$.
   (c) If $f : A \to B$ and C is a nonempty subset of A, then $f(C)$ is a nonempty subset of B.
   (d) If $f : A \to B$ is surjective and $y \in B$, then $f^{-1}(y) \in A$.
   (e) If $f : A \to B$, then A is the domain of f and B is the image of f.
   (f) A function $f : A \to B$ is surjective if $\mathit{dom}\, f = A$.
   (g) The composition of two surjective functions is always surjective.

5. Cardinality:
   (a) Two sets S and T are equipotent (or equinumerous) if there exists a bijection $f : S \to T$.
   (b) If a cardinal number is not finite, it is said to be infinite.
   (c) If a set S is finite, then S is equipotent (or equinumerous) with $\mathbb{N}_m$ for some $m \in \mathbb{N}$.