Homework 10 (Due Tues, Apr 15)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. * Let A and B be sets. Prove the following:
   (a) $\#(A \times B) = \#(B \times A)$

   Solution: We say that $\#(A\times B) = \#(B\times A)$ if there exists a bijection between $A\times B$ and $B \times A$. Let $f : A \times B \to B \times A$ be defined by $(a, b) \xrightarrow{f} (b, a)$ for $a \in A$ and $b \in B$. We will show that f is a bijection.

   First, we will show that f is injective. Suppose $f(a_1, b_1) = f(a_2, b_2)$. Then, $(b_1, a_1) = (b_2, a_2)$. So, $b_1 = b_2$ and $a_1 = a_2$. Thus, $(a_1, b_1) = (a_2, b_2)$, and f is 1-1.

   Now, we will show that f is surjective. Let $(b, a) \in B \times A$. Then, we know the element $(a, b) \in A \times B$ maps to (b, a) via f by $f(a, b) = (b, a)$. Thus, f is surjective.

   Hence, a bijection exists between $A \times B$ and $B \times A$, and $\#(A \times B) = \#(B \times A)$.

   (b) If a is an element, then $\#(\{a\} \times B) = \#B$.

   Solution: We say that $\#(\{a\} \times B) = \#B$ if there exists a bijection between $\{a\} \times B$ and B. Let $f : \{a\} \times B \to B$ be defined by $(a, b) \xrightarrow{f} b$ for $b \in B$. We will show that f is a bijection.

   First, we will show that f is injective. Suppose $f(a, b_1) = f(a, b_2)$. Then, $b_1 = b_2$. So, $(a, b_1) = (a, b_2)$, and f is 1-1.

   Now, we will show that f is surjective. Let $b \in B$. Then, we know the element $(a, b) \in \{a\} \times B$ maps to b via f by $f(a, b) = b$. Thus, f is surjective.

   Hence, a bijection exists between $\{a\} \times B$ and B, and $\#(\{a\} \times B) = \#B$.

2. * Let S and T be sets. We say that $\#S \leqslant \#T$ if there exists an injection $f : S \to T$. Is "$\leqslant$ " in this sense an equivalence relation? If so, prove it. If not, show a counterexample.

Solution: "$\leqslant$ " is reflexive and transitive; however, "$\leqslant$" is not symmetric (it is actually anti-symmetric).

To show that the relation is reflexive, we need to show that $\#S \leqslant \#S$. Let $f : S \to S$ be the identity mapping. Then, f is injective, and thus, $\#S \leqslant \#S$.

Now, we show that the relation is transitive. Let S, T, and U be sets, and suppose that $\#S \leqslant \#T$ and $\#T \leqslant \#U$. Then, there exists injective functions $f : S \to T$ and $g : T \to U$. By composing the functions, we have the function $h : S \to U$ defined by $h = g \circ f$. Since f and g are injective, we know the composition, h, is injective. Hence, $\#S \leqslant \#U$.

Now, suppose that the relation is symmetric. Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2, t_3\}$. By a simple counting argument, we can see that $\#S \leqslant \#T$. Since the relation is symmetric, there exists an injective function that maps T to S. But this is impossible (Pigeonhole Principle). Thus, "$\leqslant$" is not symmetric.