Homework 12 (Due Tues, Apr 29)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. **Statements to think about:**

   True or False. Justify your answer.

   (a) If $\{s_n\}$ is a sequence and $s_i = s_j$, then $i = j$.

   (b) If $s_n \to s$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - s| < \varepsilon$.

   (c) If for every $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $s_n < \varepsilon$, then $s_n \to 0$.

   (d) If $s_n \to k$ and $t_n \to k$ then $s_n = t_n$ for all $n \in \mathbb{N}$.

2. * Show that $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$. (Hint: Given any $\varepsilon > 0$, we have to find $N \in \mathbb{R}$ such that $n > N$ implies that $\frac{1}{\sqrt{n}} < \varepsilon$.)

Solution: Let $\varepsilon > 0$ be arbitrary. Define $N \in \mathbb{N}$ to be any integer large enough such that $N > \frac{1}{\varepsilon^2}$; notice that by simple algebraic manipulation, this implies that $\frac{1}{\sqrt{N}} < \varepsilon$. Then, for any integer $n \geqslant N$,

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{N}} < \varepsilon.$$

Therefore, we have shown that given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|\frac{1}{\sqrt{n}} - 0| < \varepsilon$. Thus $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

3. * Prove that if a sequence converges, its limit is unique. (Hint: Suppose that the sequence $\{s_n\}$ has two different limits s and t, and show that $s = t$ by showing that $|s - t| < \varepsilon$. You will need the triangle inequality: $|x + y| \leqslant |x| + |y|$.)

Solution:

Suppose that $s, t \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} s_n = t$. We will show that $s = t$, which then will prove that the limit is unique. To show that $s = t$ we will show that $|s - t| = 0$, which we will do by showing that given any $\varepsilon > 0$, $|s - t| < \varepsilon$.

To this end, take $\varepsilon > 0$ arbitrarily; note that this implies that $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$, there exists some $N_1 \in \mathbb{N}$ such that $|s_n - s| < \frac{\varepsilon}{2}$ for all $n \geqslant N_1$. Similarly, there exists some

$N_2 \in \mathbb{N}$ such that $|s_n - t| < \frac{\varepsilon}{2}$ for all $n \geqslant N_2$. Set $N = \max\{N_1, N_2\}$. Since $N \geqslant N_1$ and $N \geqslant N_2$, we have $|s_N - s| < \frac{\varepsilon}{2}$ and $|s_N - t| < \frac{\varepsilon}{2}$. Therefore,

$$\begin{aligned} |s-t| &= |s - s_N + s_N - t| \leqslant |s - s_N| + |s_N - t| \\ &= |s_N - s| + |s_N - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus we have shown that given any arbitrary $\varepsilon > 0$, $|s - t| < \varepsilon$ implying that $|s - t| = 0$, hence $s = t$.

4. * Suppose that $\lim s_n = 0$. If $\{t_n\}$ is a bounded sequence, prove that $\lim(s_n t_n) = 0$.

Solution: Suppose that $\lim s_n = 0$ and $\{t_n\}$ is a bounded sequence. Since $\{t_n\}$ is a bounded sequence, there exists an $M > 0$ such that $|t_n| < M$. Pick $\varepsilon > 0$ arbitrarily; note that this implies that $\frac{\varepsilon}{M} > 0$. Since $\lim s_n = 0$, there exists an $N \in \mathbb{N}$ such that for any integer $n \geqslant N$, $|s_n - 0| < \frac{\varepsilon}{M}$. Therefore for any integer $n \geqslant N$,

$$\begin{aligned} |s_n t_n - 0| = |s_n|\,|t_n| &= |s_n - 0|\,|t_n| \\ &< \frac{\varepsilon}{M} \cdot M = \varepsilon \end{aligned}$$

Thus, we have shown that given any arbitrary $\varepsilon > 0$, $|s_n t_n - 0| < \varepsilon$. Hence, $\lim(s_n t_n) = 0$.