Homework 2 (Due Tues, Feb 4)

Math 2710 – Spring 2014

Professor Hohn

Using the proof techniques we have learned in class, prove or give a counterexample to each statement.

1. (pg. 20 #75) The definition of the limit of a function, $\lim_{x \to a} f(x) = L$, can be expressed using quantifiers as

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x,\ (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon).$$

Use quantifiers to express the negation of this statement, which would be a definition of $\lim_{x \to a} f(x) \neq L$.

Solution: Recall that $\sim (P \implies Q)$ is logically equivalent to $P \wedge \sim Q$. The negation of the statement above is:

$$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists x \neq a,\ (0 < |x - a| < \delta \text{ and } |f(x) - L| \geqslant \epsilon)\,.$$

2. (pg. 50 #10) If $ac \mid bc$ and $c \neq 0$, prove that $a \mid b$.

Solution:

Proof. Let $ac \mid bc$, $c \neq 0$. Then, for some $q \in \mathbb{Z}$,

$$\begin{aligned} bc &= (ac)q \\ 0 &= acq - bc \\ 0 &= c(aq - b)\,, \end{aligned}$$

where $(aq - b) \in \mathbb{Z}$. Then, $c = 0$ or $aq - b = 0$. By assumption, $c \neq 0$, and so $b = aq$. Hence, $a \mid b$. □

3. (pg. 50 #99, revised) Prove that if $a \mid b$, then $a^2 \mid b^2$.

Solution:

Proof. Suppose $a \mid b$. Then, $b = aq$ for $q \in \mathbb{Z}$, and

$$\begin{aligned} b^2 &= b(aq) \\ &= (aq)(aq) \\ &= a^2 q^2 . \end{aligned}$$

Thus, $a^2 \mid b^2$. □