Homework 3 (Due Tues, Feb 11)

Math 2710 – Spring 2014

Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. (pg. 50 #73) Prove that $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

Solution:

Proof. Suppose $e \in \{ax + by \mid x, y \in \mathbb{Z}\}$. Then, $e = ax + by$ for some $x, y \in \mathbb{Z}$. By the Linear Diophantine Theorem, $\gcd(a, b) \mid e$. Hence, $\exists n \in \mathbb{Z}$ such that $e = n \cdot \gcd(a, b)$. Then, $e \in \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$. So, $\{ax + by \mid x, y \in \mathbb{Z}\} \subseteq \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

Let $f \in \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$. Then, $f = n \cdot \gcd(a, b)$ for some $n \in \mathbb{Z}$. By the Euclidean Algorithm, $\exists x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$. Thus, $f = n \cdot (ax + by)$ and $f = a \cdot nx + b \cdot ny$. Let $\hat{x} = nx$ and $\hat{y} = ny$. Then, $f = a\hat{x} + b\hat{y}$ for some $\hat{x}, \hat{y} \in \mathbb{Z}$. Hence, $f \in \{ax + by \mid x, y \in \mathbb{Z}\}$. Therefore, the set $\{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\} \subseteq \{ax + by \mid x, y \in \mathbb{Z}\}$, and the sets are equal. □

2. (pg. 50 #83) Let a, b, c be nonzero integers. Their *greatest common divisor* $\gcd(a, b, c)$ is the largest positive integer that divides all of them. Prove that

$$\gcd(a, b, c) = \gcd(a, \gcd(b, c))\,.$$

Solution:

Proof. Let e be the $\gcd(a, \gcd(b, c))$ and let $d = \gcd(b, c)$. By the Euclidean Algorithm, $e = a \cdot x + d \cdot w$ for $x, w \in \mathbb{Z}$. In addition, $d = b \cdot y + c \cdot z$ for $y, z \in \mathbb{Z}$. Then, $e = a \cdot x + b \cdot yw + c \cdot zw$. Since $e \mid a$ and $e \mid d$, $e \mid b$ and $e \mid c$. Hence, e is a common divisor of a, b, c.

Suppose e is not the greatest common divisor of a, b, and c. Then, there exists an $\hat{e} = \gcd(a, b, c)$ where $\hat{e} > e$. By definition of the gcd, $\hat{e} \mid a$, $\hat{e} \mid b$, and $\hat{e} \mid c$. Thus, $\hat{e} \mid (a \cdot x + b \cdot yw + c \cdot zw)$ and $\hat{e} \mid e$. Then, $e = \hat{e} \cdot n$ for some positive $n \in \mathbb{Z}$. Because $e < \hat{e}$ and $e = \hat{e} \cdot n$, it follows that $\hat{e} \cdot n < \hat{e}$ ↯. Thus, $e = \gcd(a, b, c) = \gcd(a, \gcd(b, c))$.

□

3. (pg. 50 #102) Prove or give a counter example.

If $\gcd(a, b) = 1$ and $ax + by = c$ has a positive integer solution, then so does $ax + by = d$ when $d > c$.

Solution:

Counterexample: Let $a = 2$, $b = 3$. The $\gcd(2, 3) = 1$ and $2x + 3y = 5$ has the positive integer solution $(1, 1)$. However, the equation $2x + 3y = 6$ does not have a solution.