Homework 5 (Due Tues, Feb 25)

Math 2710 – Spring 2014

Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. (pg. 50 #97) Let a and b be integers greater than 1, and let $e = \operatorname{lcm}(a, b)$. Prove that

$$0 < \frac{1}{a} + \frac{1}{b} - \frac{1}{e} < 1\,.$$

> **Solution:** Let $e = \operatorname{lcm}(a, b)$, $a, b \in \mathbb{Z}$ and a, b greater than 1. Then,
>
> $$\begin{aligned} \frac{1}{a} + \frac{1}{b} - \frac{1}{e} &= \frac{1}{a} + \frac{1}{b} - \frac{1}{\frac{ab}{\gcd(a,b)}} \\ &= \frac{1}{a} + \frac{1}{b} - \frac{\gcd(a,b)}{ab} \\ &= \frac{b}{ab} + \frac{a}{ab} - \frac{\gcd(a,b)}{ab} \end{aligned}$$
>
> Since $\gcd(a, b) \mid a$ and $\gcd(a, b) \mid b$, $\gcd(a, b) \leqslant a$ and $\gcd(a, b) \leqslant b$. Therefore,
>
> $$0 < a + b - \max\{a, b\} \leqslant a + b - \gcd(a, b) < a + b \leqslant ab\,.$$
>
> Hence,
>
> $$0 < a + b - \gcd(a, b) < ab\,,$$
>
> and
>
> $$0 < \frac{1}{a} + \frac{1}{b} - \frac{1}{e} < 1\,.$$

2. (pg. 50 #100) Prove or give a counterexample. $\gcd(a, b) = \gcd(a + b, \operatorname{lcm}(a, b))\,.$

> **Solution:** Let $d = \gcd(a, b)$. By the definition of gcd, $a = d \cdot p$ and $b = d \cdot q$ where p, q are relatively prime. The $\operatorname{lcm}(a, b) = \operatorname{lcm}(dp, dq)$, and by definition of lcm, $\operatorname{lcm}(a, b) = dpq$. So,
>
> $$\begin{aligned} \gcd(a + b, \operatorname{lcm}(a, b)) &= \gcd(dp + dq, \operatorname{lcm}(dp, dq)) \\ &= \gcd(d(p + q), dpq) \\ &= d \cdot \gcd(p + q, pq) \text{ see \# 11 hwk}\,. \end{aligned}$$

Since p, q are relatively prime, $\gcd(p+q, pq) = 1$. Hence, $\gcd(a+b, \operatorname{lcm}(a,b)) = d$.

3. (pg. 50 #101) Prove or give a counterexample. $\operatorname{lcm}(\gcd(a,b), \gcd(a,c)) = \gcd(a, \operatorname{lcm}(b,c))$.

Solution: Let $a = p_1^{a_1} \dots p_n^{a_n}$, $b = p_1^{b_1} \dots p_n^{b_n}$, and $c = p_1^{c_1} \dots p_n^{c_n}$ where p_i is prime. Then, $\gcd(a,b) = d = p_1^{d_1} \dots p_n^{d_n}$ where $d_i = \min(a_i, b_i)$, and $\gcd(a,c) = e = p_1^{e_1} \dots p_n^{e_n}$ where $e_i = \min(a_i, c_i)$. The $\operatorname{lcm}(d,e) = f = p_1^{f_1} \dots p_n^{f_n}$ where $f_i = \max(d_i, e_i) = \max(\min(a_i, b_i), \min(a_i, c_i))$. In a similar fashion, $\gcd(a, \operatorname{lcm}(b,c)) = g = p_1^{g_1} \dots p_n^{g_n}$ where $g_i = \min(a_i, \max(b_i, c_i))$. To show $f = g$, we show that $f_i = g_i$ for all i.

Let $f_i = \max(\min(a_i, b_i), \min(a_i, c_i))$ and $g_i = \min(a_i, \max(b_i, c_i))$. Suppose $c_i > b_i > a_i$. Then, $f_i = a_i$ and $g_i = a_i$ and $f = g$. This is also true for $c_i > a_i > b_i$.

Suppose $b_i > c_i > a_i$. Then, $f_i = a_i$ and $g_i = a_i$ and $f = g$. This is also true for $b_i > a_i > c_i$.

Suppose $a_i > b_i > c_i$. Then, $f_i = b_i$ and $g_i = b_i$ and $f = g$.

Suppose $a_i > c_i > b_i$. Then, $f_i = c_i$ and $g_i = c_i$ and $f = g$.

Hence, $f = g$ and $\operatorname{lcm}(\gcd(a,b), \gcd(a,c)) = \gcd(a, \operatorname{lcm}(b,c))$.