Homework 6 (Due Tues, March 11)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. Let A, B, C be sets. Prove the following:

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   Solution:

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\iff x \in (A \cap B) \text{ or } x \in (A \cap C) \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

   (b) $A - (B \cup C) = (A - B) \cap (A - C)$

   Solution:

$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin (B \cup C) \\ &\iff x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\iff x \in (A - B) \text{ and } x \in (A - C) \\ &\iff x \in (A - B) \cap (A - C) \end{aligned}$$

   (c) $(A \cup B)' = A' \cap B'$

   Solution:

$$\begin{aligned} x \in (A \cup B)' &\iff x \notin (A \text{ or } B) \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A' \text{ and } x \in B' \\ &\iff x \in A' \cap B' \end{aligned}$$

2. Let A be a set and $\{B_i\}_{i \in I}$ an indexed family of sets. Prove the following:

$$\left(\bigcup_{i \in I} B_i \right) \cap A = \bigcup_{i \in I} (B_i \cap A).$$

Solution:

$$\begin{aligned}
x \in \left(\bigcup_{i\in I} B_i\right) \cap A &\iff x \in \left(\bigcup_{i\in I} B_i\right) \text{ and } x \in A \\
&\iff (\exists i \in I)(x \in B_i) \text{ and } x \in A \\
&\iff (\exists i \in I)(x \in B_i \text{ and } x \in A) \\
&\iff (\exists i \in I)(x \in B_i \cap A) \\
&\iff x \in \bigcup_{i\in I} (B_i \cap A)
\end{aligned}$$

3. * Let A be a set and let $\{B_i\}_{i\in I}$ be an indexed family of sets. Prove the following is true.

$$A - \bigcup_{i\in I} B_i = \bigcap_{i\in I} (A - B_i)$$

Solution:

Proof.

$$\begin{aligned}
x \in A - \bigcup_{i\in I} B_i &\iff x \in A \text{ and } x \notin \bigcup_{i\in I} B_i \\
&\iff x \in A \text{ and } (x \notin B_i)(\forall i \in I) \\
&\iff (\forall i \in I)(x \in A \text{ and } x \notin B_i) \\
&\iff (\forall i \in I)(x \in (A - B_i)) \\
&\iff x \in \bigcap_{i\in I} (A - B_i)
\end{aligned}$$

□

4. Show that if $A \subset B$ then $\mathcal{P}(A) \subset \mathcal{P}(B)$. ($\mathcal{P}(A)$ is the power set of A.)

Solution: Let $X \in \mathcal{P}(A)$. By definition of the power set, $X \subseteq A$. By assumption $A \subset B$, so since $X \subseteq A$ and $A \subset B$, it follows that $X \subset B$. Hence, $X \in \mathcal{P}(B)$. Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. To show that $\mathcal{P}(A) \subset \mathcal{P}(B)$, we will show that $\mathcal{P}(A) \neq \mathcal{P}(B)$.

Since $A \subset B$, $\exists x \in B - A$. Hence, $\{x\} \subseteq B$ and $\{x\} \nsubseteq A$. Therefore, $\{x\} \subset \mathcal{P}(B)$ and $\{x\} \not\subset \mathcal{P}(A)$. Thus, $\mathcal{P}(A) \neq \mathcal{P}(B)$, and $\mathcal{P}(A) \subset \mathcal{P}(B)$.

5. Let $S = \{1, 2, 3\}$. In each case, give an example of a relation R on S that has the stated properties.

   (a) R is not symmetric, not reflexive, and not transitive.

Solution: Define the relation R on the set S by the following:

$$R = \{(1,2), (3,3), (2,3)\}.$$

R is not reflexive since R does not have the relations $(1,1)$ and $(2,2)$. R is not symmetric since $(2,1)$ and $(3,2)$ do not exist in R. R is not transitive because $(1,2)$ and $(2,3)$ exist in R, but $(1,3)$ does not.

(b) R is transitive and reflexive, but not symmetric.

Solution: Let R be the relation "$\leqslant$" – less than or equal to – on the set S. It is always true that $x \leqslant x$, so R is reflexive. If $x \leqslant y$ and $y \leqslant z$, then it follows that $x \leqslant y \leqslant z$ and $x \leqslant z$. Thus, R is transitive. However, if $x \leqslant y$ is does not follow that $y \leqslant x$. For example, let $x = 1$, $y = 2$. Then, $1 \leqslant 2$, but $2 \nleqslant 1$. Thus, R is not symmetric.

6. * A relation R is *antisymmetric* if xRy and yRx together imply that $x = y$. A relation R on S is a partial ordering if R is reflexive, antisymmetric, and transitive. For example, the relation "$\leqslant$" on $\mathbb{R}$ is a partial ordering . Show that each of the following is a partial ordering.

(a) The inclusion relation "$\subseteq$" on the power set of a given set A.

Solution: Let A be a set, and let $\mathcal{P}(A)$ represent the power set of A. Define the relation $\sim$ to be the inclusion relation on the power set of A. To show $\sim$ is a partial ordering, we will show that $\sim$ is reflexive, antisymmetric, and transitive.

Let $X, Y, Z \in \mathcal{P}(A)$. Since $X \subseteq X$, X is related to itself, and $\sim$ is reflexive.

Suppose $X \sim Y$ and $Y \sim X$. Then, $X \subseteq Y$ and $Y \subseteq X$, and $X = Y$. Hence, $\sim$ is antisymmetric.

Suppose $X \sim Y$ and $Y \sim Z$. Then, $X \subseteq Y$ and $Y \subseteq Z$. So $X \subseteq Z$ and $X \sim Z$. Hence, $\sim$ is transitive.

Thus, $\sim$ is a partial ordering.

(b) The divisibility relation on $\mathbb{N}$. (If $a, b \in \mathbb{N}$, define $a \mid b$ to mean that $b = a \cdot q$ for some $q \in \mathbb{N}$.)

Solution: Let R be the divisibility relation on $\mathbb{N}$. To prove that R is a partial ordering, we will show R is reflexive, antisymmetric, and transitive.

Let $a, b, c \in \mathbb{N}$. Then, $a = a \cdot 1$ and $a \mid a$.

Suppose $a \mid b$ and $b \mid a$. Then, $b = a \cdot q$ and $a = b \cdot p$ for some $p, q \in \mathbb{N}$. So, $a = (a \cdot q) \cdot p$ and $p \cdot q = 1$ which implies $p = q = 1$. Thus, $a = b$.

Suppose $a \mid b$ and $b \mid c$. Then, $\exists p, q \in \mathbb{N}$ such that $b = a \cdot p$ and $c = b \cdot q$. So, $c = (a \cdot p) \cdot q$ and since $pq \in \mathbb{N}$, $c = a \cdot pq$. Thus, $a \mid c$ and R is transitive.

Therefore, R is a partial ordering.

7. Determine each of the following sets.

(a) $\mathcal{P}(\{2\})$

Solution: Let $A = \{2\}$. Then,

$$\begin{aligned}\mathcal{P}(A) &= \{\varnothing, A\} \\ &= \{\varnothing, \{2\}\}.\end{aligned}$$

(b) $\mathcal{P}(\mathcal{P}(\{2\}))$

Solution: Let $A = \{2\}, B = \varnothing$. Then,

$$\begin{aligned}\mathcal{P}(\mathcal{P}(A)) &= \mathcal{P}(\{B, A\}) \\ &= \Big\{\varnothing, \{A\}, \{B\}, \{A, B\}\Big\} \\ &= \Big\{\varnothing, \big\{\{2\}\big\}, \{\varnothing\}, \big\{\{2\}, \varnothing\big\}\Big\}.\end{aligned}$$

(c) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{2\})))$

Solution: Let $A = \varnothing, B = \big\{\{2\}\big\}, C = \{\varnothing\}, D = \big\{\{2\}, \varnothing\big\}$. Then,

$$\begin{aligned}\mathcal{P}(\mathcal{P}(\mathcal{P}(A))) &= \mathcal{P}(\{A, B, C, D\}) \\ &= \Big\{\varnothing, \{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}, \\ &\quad \{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}, \{A, B, C, D\}\Big\} \\ &= \Bigg\{\varnothing, \{\varnothing\}, \Big\{\big\{\{2\}\big\}\Big\}, \big\{\{\varnothing\}\big\}, \Big\{\big\{\{2\}, \varnothing\big\}\Big\}, \Big\{\varnothing, \big\{\{2\}\big\}\Big\}, \big\{\varnothing, \{\varnothing\}\big\}, \Big\{\varnothing, \big\{\{2\}, \varnothing\big\}\Big\}, \\ &\quad \Big\{\big\{\{2\}\big\}, \{\varnothing\}\Big\}, \Big\{\big\{\{2\}\big\}, \big\{\{2\}, \varnothing\big\}\Big\}, \Big\{\{\varnothing\}, \big\{\{2\}, \varnothing\big\}\Big\}, \Big\{\varnothing, \big\{\{2\}\big\}, \{\varnothing\}\Big\}, \\ &\quad \Big\{\varnothing, \big\{\{2\}\big\}, \big\{\{2\}, \varnothing\big\}\Big\}, \Big\{\varnothing, \{\varnothing\}, \big\{\{2\}, \varnothing\big\}\Big\}, \Big\{\big\{\{2\}\big\}, \{\varnothing\}, \big\{\{2\}, \varnothing\big\}\Big\}, \\ &\quad \Big\{\varnothing, \big\{\{2\}\big\}, \{\varnothing\}, \big\{\{2\}, \varnothing\big\}\Big\}\Bigg\}\end{aligned}$$

8. Given any two sets S and T, the *Cartesian product* of S and T is the new set $S \times T$ defined by

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

If S and T are sets, $A \subseteq S$ and $B \subseteq T$, prove that $A \times B \subseteq S \times T$.

Solution: Let A and B be sets and let $(x, y) \in A \times B$. By definition, $x \in A$ and $y \in B$. By assumption, $A \subseteq S$, so $x \in S$. Likewise, $B \subseteq T$ and $y \in T$. Hence, $(x, y) \in S \times T$ and $A \times B \subseteq S \times T$.

9. * Prove that A and B are disjoint if and only if $A \subseteq B'$.

Solution: ($\Longrightarrow$) Suppose $A \cap B = \varnothing$. Let $x \in A$. Since $A \cap B = \varnothing$, if $x \in A$, $x \notin B$. Thus, $x \in B'$, and $A \subseteq B'$.

($\Longleftarrow$) Suppose $A \subseteq B'$ and suppose $x \in A \cap B$. Then, $x \in A$ and $x \in B$. By assumption, if $x \in A$, $x \notin B$. Contradiction $\lightning$. Thus, $\nexists x \in A \cap B$, and $A \cap B = \varnothing$.

10. Prove or give a counterexample to the assertion $A \cup (B - A) = B$.

Solution: Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then, $B - A = \{4, 5\}$.

$$A \cup (B - A) = \{1, 2, 3, 4, 5\} \neq \{3, 4, 5\} = B.$$