# Homework 7 (Due Tues, March 25)

**Math 2710 – Spring 2014**

**Professor Hohn**

Using the proof techniques we have learned in class, prove each statement.

1. * Prove that the inequality $n^2 \geqslant n$ holds *for every integer*.

**Solution:** Let $n \in \mathbb{Z}$. If $n \leqslant 0$, then $n^2 \geqslant n$ for all $n$ since $n^2 \geqslant 0 \geqslant n$. Hence, the inequality holds for all negative integers and $n = 0$. Now, we will prove the inequality holds for all $n \geqslant 1$ via induction on $n$.

First, we will prove the base case, $P(1)$.

$$1^2 \geqslant 1$$

and $P(1)$ is true.

Now, assume $P(k)$ is true. That is, $k^2 \geqslant k$. We will show that $P(k+1)$ is true.

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &\geqslant k + 2k + 1 \qquad \text{by } P(k) \\ &= 3k + 1 \\ &\geqslant k + 1 \qquad \text{since } k \geqslant 1\end{aligned}$$

Thus, $P(k) \implies P(k+1)$ is true, and by the Principle of Mathematical Induction, $n^2 \geqslant n$ for all $n \in \mathbb{N}$.

2. (pg.105 #33) Use induction to prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \geqslant 1 + \frac{n}{2}.$$

**Solution:** (via induction on $n$)

First, we will show the base case, $P(0)$, is true.

$$\frac{1}{2^0} = \frac{1}{1} \geqslant 1 + \frac{0}{2}$$

and $P(0)$ is true.

Now, assume $P(k)$ is true. That is, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}\geqslant 1+\frac{k}{2}$. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\ldots+\frac{1}{2^k+2^k} &\geqslant 1+\frac{k}{2}+\frac{1}{2^k+1}+\ldots+\frac{1}{2^k+2^k} \text{ (by } P(k))\\
&\geqslant 1+\frac{k}{2}+\frac{1}{2^k+2^k}+\ldots+\frac{1}{2^k+2^k}\\
&= 1+\frac{k}{2}+\frac{2^k}{2^k+2^k}\\
&= 1+\frac{k}{2}+\frac{2^k}{2^{k+1}}\\
&= 1+\frac{k}{2}+\frac{1}{2}\\
&= 1+\frac{k+1}{2}
\end{aligned}$$

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all $n\in\mathbb{Z}$ where $n\geqslant 0$.

3. (pg. 108 #63) If $a=\dfrac{1+\sqrt{5}}{2}$ and $b=\dfrac{1-\sqrt{5}}{2}$, prove that $f_n=\dfrac{a^n-b^n}{\sqrt{5}}$ for all $n\in\mathbb{N}$.

**Solution:** (via induction – v4)

The definition of $f_n$ involves the recursion formula $f_n=f_{n-1}+f_{n-2}$. For this reason, we will individually verify $P(1)$, $P(2)$, and $P(3)$.

Base case – $P(1)$ or $f_1$:

$$\frac{a-b}{\sqrt{5}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}=\frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}}=1=f_1.$$

$P(2)$ or $f_2$:

$$\frac{a^2-b^2}{\sqrt{5}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}}=\frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}}=1=f_2.$$

$P(3)$:

$$\frac{a^3-b^3}{\sqrt{5}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^3-\left(\frac{1-\sqrt{5}}{2}\right)^3}{\sqrt{5}}=\frac{\frac{16\sqrt{5}}{8}}{\sqrt{5}}=2=f_3.$$

Now, assume for all $3\leqslant t\leqslant k$, $P(t)$ is true. We will show that $P(k+1)$ is true.

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \qquad \text{(by the recursion definition of } f_{k+1}\text{)} \\
&= \frac{a^k - b^k}{\sqrt{5}} + \frac{a^{k-1} - b^{k-1}}{\sqrt{5}} \\
&= \frac{a^{k-1}(a+1) - b^{k-1}(b+1)}{\sqrt{5}} \\
&= \frac{a^{k-1}(a^2) - b^{k-1}(b^2)}{\sqrt{5}} \qquad \text{(since } a+1 = a^2 \text{ and } b+1 = b^2\text{)} \\
&= \frac{a^{k+1} - b^{k+1}}{\sqrt{5}}
\end{aligned}
$$

So, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.