# Homework 7 (Due Tues, March 25)

**Math 2710 – Spring 2014**
**Professor Hohn**

Using the proof techniques we have learned in class, prove each statement.

1. * Prove that the inequality $n^2 \geqslant n$ holds *for every integer*.
2. Prove that for every integer $n \geqslant 0$, the number $n^4 - 4n^2$ is divisible by 3.

**Solution:** (via induction)
Base case - $P(0)$:

$$0^4 - 4 \cdot 0^2 = 0 = 0 \cdot 3$$

is divisible by 3. Thus, $P$ holds for $n = 0$.
Now, assume $P(k)$. Then, for some $a \in \mathbb{Z}$, $k^4 - 4k^2 = 3 \cdot a$. We will show $P(k+1)$ is true.

$$\begin{aligned}(k+1)^4 - 4(k+1)^2 &= (k^4 + 4k^3 + 6k^2 + 4k + 1) - 4(k^2 + 2k + 1) \\ &= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3 \\ &= 3 \cdot a - 3 + 4k^3 + 6k^2 - 4k\end{aligned}$$

We will show by induction that $\forall k \geqslant 0$, $3 \mid 4k^3 + 6k^2 - 4k$.
Base case - $\hat{P}(0)$:

$$4 \cdot 0^3 + 6 \cdot 0^2 - 4 \cdot 0 = 0 = 3 \cdot 0$$

Now, assume $\hat{P}(i)$. Then, for some $b \in \mathbb{Z}$, $4i^3 + 6i^2 - 4i = 3 \cdot b$.

$$\begin{aligned}4(i+1)^3 + 6(i+1)^2 - 4(i+1) &= 4(i^3 + 3i^2 + 3i + 1) + 6(i^2 + 2i + 1) - 4i - 4 \\ &= 4i^3 + 6i^2 - 4i + 12i^2 + 24i + 6 \\ &= 3 \cdot b + 3(4i^2 + 8i + 2)\end{aligned}$$

which is divisible by 3. Hence, for all $k \geqslant 0$, $3 \mid 4k^3 + 6k^2 - 4k$.
Therefore, $3 \mid (k+1)^4 - 4(k+1)^2$ and $P(k+1)$ is true. Hence, for every integer $n \geqslant 0$, the number $n^4 - 4n^2$ is divisible by 3.

*Another proof without induction:* We want to show that for every integer $n \geqslant 0$, the number $n^4 - 4n^2$ is divisible by 3. If $3 \mid n$, then $3 \mid n^4 - 4n^2$ and we're done. Suppose $3 \nmid n$. Then, $n = 3p + 1$ or $n = 3p + 2$ for some $p \in \mathbb{Z}$. If $n = 3p + 1$, then

$$\begin{aligned}n^4 - 4n^2 &= n^2(n-2)(n+2) \\ &= (3p+1)^2(3p-1)(3p+3) \\ &= 3 \cdot (3p+1)^2(3p-1)(p+1)\end{aligned}$$

and $3 \mid n^4 - 4n^2$. Now, suppose $n = 3p + 2$. By the same argument,

$$\begin{aligned} n^4 - 4n^2 &= n^2(n-2)(n+2) \\ &= (3p+1)^2(3p)(3p+4) \\ &= 3 \cdot (3p+1)^2(p)(3p+4) \end{aligned}$$

and $3 \mid n^4 - 4n^2$. Hence, for all $n \geqslant 0$, $n^4 - 4n^2$ is divisible by 3.

3. Prove that $2^n > n^3$ for every integer $n \geqslant 10$.

**Solution:** (via induction)

Base case - $P(10)$:

$$10^3 = 1000 < 1024 = 2^{10}$$

Now, assume $P(k)$. That is, assume $2^k > k^3$. We will show that $P(k+1)$ is true.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^3 \qquad \text{(by } P(k)\text{)} \end{aligned}$$

We will show that $2k^3 > (k+1)^3$ which will prove that $P(k+1)$ is true. Because we can rewrite $2k^3 > (k+1)^3$ in the following way

$$\begin{aligned} 2k^3 &> (k+1)^3 \\ 2k^3 &> k^3 + 3k^2 + 3k + 1 \\ k^3 - 3k^2 - 3k &> 1 \\ k(k^2 - 3k - 3) &> 1\,, \end{aligned}$$

it suffices to show that $k(k^2-3k-3) > 1$. Since $k \geqslant 10$, it suffices to show that $k^2-3k-3 > 1$ for all $k \geqslant 10$.

$$\begin{aligned} k^2 - 3k - 3 &> 1 \\ k^2 - 3k - 4 &> 0 \\ (k-4)(k+1) &> 0 \end{aligned}$$

For all $k \geqslant 10$, $(k-4)(k+1) > 0$, and thus, $2k^3 > (k+1)^3$. Hence, $P(k+1)$ is true. Therefore, $2^n > n^3$ for every integer $n \geqslant 10$.

4. For each $i \in \mathbb{N}$, let $a_i = 3^{i-2}$. Evaluate

   (a) $\displaystyle\sum_{i=1}^{5} a_i$,

**Solution:**

$$\begin{aligned}\sum_{i=1}^{5} a_i &= \sum_{i=1}^{5} 3^{i-2} \\ &= 3^{-1} + 3^0 + 3^1 + 3^2 + 3^3 + 3^4 \\ &= \frac{1}{3} + 1 + 3 + 9 + 27 + 81 \\ &= \frac{122}{3}\end{aligned}$$

(b) $\prod_{i=1}^{5} a_i$.

**Solution:**

$$\begin{aligned}\prod_{i=1}^{5} a_i &= \prod_{i=1}^{5} 3^{i-2} \\ &= 3^{-1} \cdot 3^0 \cdot 3 \cdot 3^2 \cdot 3^3 \cdot 3^4 \\ &= 3^9\end{aligned}$$