Homework 8 (Due Tues, Apr 1)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. For functions whose domains are sets of real numbers, it is common practice to use a formula to describe a function's pairing rule with the understanding that the domain of the function is the set of all real numbers for which the formula gives a unique real number unless further restrictions are imposed. For example, the function f given by $f(x) = \sqrt{x-3}$ has domain $x \in \mathbb{R} \mid x \geqslant 3$. In each of the following, determine the domains of f and g, and then use the definition of *equal* to show whether f and g are equal.
   (a) $f(x) = 1$, $g(x) = \dfrac{x-5}{x-5}$
   (b) $f(x) = \sqrt{x}$, $g(x) = \sqrt{|x|}$
   (c) $f(x) = |x|$, $g(x) = \sqrt{x^2}$
   (d) $f(x) = x^2 - x - 6$, $g(x) = (x-4)(x+3) + 6$
   (e) $f(x) = x^2$, $g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

2. Suppose a function $f : A \to B$ is given. Define a relation $\sim$ on A as follows:

$$a_1 \sim a_2 \iff f(a_1) = f(a_2)$$

   Prove that $\sim$ is an *equivalence relation* on A.

3. * If A and B are sets, let B^A denote the set of all functions from A to B.
   (a) Determine the set $\{1,2\}^{\{1,2\}}$. That is, list it's members explicitly.
   (b) Show that if A, B, C are sets and $A \subseteq B$, then $A^C \subseteq B^C$.
   (c) Show that the sets $\{1,2\}^{\{1,2\}}$ and $\{1,2\}^{\{1,2,3\}}$ are disjoint.
   (d) Generalize part (c) by showing that if $B \neq C$ then $A^B \cap A^C = \varnothing$.
   (e) Show that if A is a nonempty set, then $\varnothing^A = \varnothing$.
   (f) Show that if B is any set then $B^{\varnothing} = \{\varnothing\}$.

4. In each case, state without proof whether the given function is injective, surjective, bijective.
   (a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^3$.
   (b) $g : \{1,2,3\} \to \{-2,5,6\}$ given by $g = \{(1,5),(2,-2),(3,6)\}$.
   (c) $h : \{1,2,3\} \to \{-2,5,6\}$ given by $h = \{(3,5),(2,-2),(1,5)\}$.

5. Let S and T by sets with three elements and two elements, respectively. In each case, state the answer and justify briefly.
   (a) How many functions are there from S to T?
   (b) How many injections are there from S to T?
   (c) How many injections are there from T to S?
   (d) How many surjections are there from S to T?
   (e) How many surjections are there from T to S?
   (f) Guess the answers to (a) and (b) if S has m elements and T has n elements.

6. * A function $f : A \to B$ is a subset of $A \times B$ and hence a subset of $(A \cup B) \times (A \cup B)$. By our definition of relation, f is a relation on $A \cup B$. Therefore, f has an inverse relation
$$f^{-1} = \{(b, a) \mid (a, b) \in f\}.$$
   (a) Write the inverses of
      i. $g : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $g = \{(1, 5), (2, -2), (3, 6)\}$.
      ii. $h : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $h = \{(3, 5), (2, -2), (1, 5)\}$.
   (b) Show that if f is not injective, then f^{-1} is not a function.
   (c) Show that if f is injective but not surjective, then f^{-1} is a function whose domain is a proper subset of B.

7. Let a and b be real numbers. Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = ax+b$.
   (a) Under what conditions on a and b is f a bijection from $\mathbb{R}$ to $\mathbb{R}$?
   (b) Under what conditions on a and b is the restriction $f{\restriction}_{\mathbb{Z}}$ a bijection from $\mathbb{Z}$ to $\mathbb{Z}$?
   (c) Under what conditions on a and b is the restriction $f{\restriction}_{\mathbb{N}}$ a bijection from $\mathbb{N}$ to $\mathbb{N}$?

8. Define the 1-tuple $(a_1) = \{a_1\}$. Define the 2-tuple (a_1, a_2) to be the ordered pair (a_1, a_2). In general, if the k-tuple $(a_1, a_2, \ldots, a_k)$ has been defined, define the $(k+1)$-tuple $(a_1, a_2, \ldots, a_k, a_{k+1})$ to be the ordered pair
$$\Big((a_1, \ldots, a_k), a_{k+1}\Big).$$
Prove by induction on n that this recursive definition of n-tuple retains the following essential property:
$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \iff a_i = b_i \text{ for each } i,\ 1 \leqslant i \leqslant n.$$

9. Information transmitted electronically is usually encoded in the form of n-tuples of 0s and 1s, often called **n-bit strings**. For brevity, commas and parentheses are omitted; thus, 01101011 is an 8-bit string. A **code of length** n is a set of such strings, and the members of code are called codewords. To aid in the detection and correction of errors, it is useful to have different codewords differ in more that one coordinate. Define the **distance** between two n-bit strings to be the number of coordinates in which they differ. More formally, let $\mathbb{Z}_2^n$ denote the set of n-bit strings, and define a function $d : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}$ by
$$d(a_1a_2 \ldots a_n, b_1b_2 \ldots b_n) = \sum_{i=1}^{n} |a_i - b_i|$$
where $a_i, b_i \in \{0, 1\}$ for each i. For example, $d(1101, 1001) = 1$. Prove that the function d has the following properties:

(a) $d(s_1, s_2) \geqslant 0$

(b) $d(s_1, s_2) = 0 \iff s_1 = s_2$

(c) $d(s_1, s_2) = d(s_2, s_1)$

(d) For all $s_1, s_2, s_3 \in \mathbb{Z}_2^n$,

$$d(s_1, s_3) \leqslant d(s_1, s_2) + d(s_2, s_3) \qquad \text{(the } \textbf{triangle inequality}\text{)}$$

Suggestion: Use induction on n, starting with $n = 1$.

10. * Suppose $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are surjections. Prove that there is a surjection $h : \mathbb{N} \to A \cup B$. [Suggestion: Consider the list $f(1), g(1), f(2), g(2), \ldots$]