Homework 8 (Due Tues, Apr 1)

Math 2710 – Spring 2014

Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. For functions whose domains are sets of real numbers, it is common practice to use a formula to describe a function's pairing rule with the understanding that the domain of the function is the set of all real numbers for which the formula gives a unique real number unless further restrictions are imposed. For example, the function f given by $f(x) = \sqrt{x-3}$ has domain $x \in \mathbb{R} \mid x \geqslant 3$. In each of the following, determine the domains of f and g, and then use the definition of *equal* to show whether f and g are equal.

   (a) $f(x) = 1$, $g(x) = \dfrac{x-5}{x-5}$

   Solution: $D_f = \mathbb{R}$ and $D_g = \mathbb{R} - \{5\}$. Thus, $f \neq g$.

   (b) $f(x) = \sqrt{x}$, $g(x) = \sqrt{|x|}$

   Solution: $D_f = \{x \in \mathbb{R} \mid x \geqslant 0\}$ and $D_g = \mathbb{R}$. Thus, $f \neq g$.

   (c) $f(x) = |x|$, $g(x) = \sqrt{x^2}$

   Solution: $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$, and $\forall x \in \mathbb{R}$, $f(x) = g(x)$. Thus, $f = g$.

   (d) $f(x) = x^2 - x - 6$, $g(x) = (x-4)(x+3) + 6$

   Solution: $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$, and $\forall x \in \mathbb{R}$, $f(x) = g(x)$. Thus, $f = g$.

   (e) $f(x) = x^2$, $g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

   Solution: $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$, but $\forall x \in \mathbb{R}$, $f(x) \neq g(x)$ ($f(\pi) = \pi^2$ and $g(\pi) = 0$). Thus, $f \neq g$.

2. Suppose a function $f : A \to B$ is given. Define a relation $\sim$ on A as follows:

$$a_1 \sim a_2 \iff f(a_1) = f(a_2)$$

   Prove that $\sim$ is an *equivalence relation* on A.

Solution: $\sim$ is an equivalence relation if $\sim$ is reflexive, symmetric, and transitive.

(Reflexive) Let $a \in A$. Since equality is reflexive, $f(a) = f(a)$ for all $a \in A$, and hence $a \sim a$. Therefore, $\sim$ is reflexive.

(Symmetric) Let $a, b \in A$. If $a \sim b$, then $f(a) = f(b)$. Since equality is symmetric, $f(b) = f(a)$ and hence $b \sim a$. Hence, $\sim$ is symmetric.

(Transitive) Let $a, b, c \in A$. Suppose $a \sim b$ and $b \sim c$. Then, $f(a) = f(b)$ and $f(b) = f(c)$. Since equality is transitive, this implies that $f(a) = f(c)$ and hence $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transtive, $\sim$ is an equivalence relation.

3. * If A and B are sets, let B^A denote the set of all functions from A to B.

(a) Determine the set $\{1,2\}^{\{1,2\}}$. That is, list it's members explicitly.

Solution: The members of the set $\{1,2\}^{\{1,2\}}$ are all functions that map from $\{1,2\}$ to $\{1,2\}$. Let $f_1 = \{(1,1),(2,1)\}$, $f_2 = \{(1,1),(2,2)\}$, $f_3 = \{(1,2),(2,1)\}$, and $f_4 = \{(1,2),(2,2)\}$. Then, $\{1,2\}^{\{1,2\}} = \{f_1, f_2, f_3, f_4\}$.

(b) Show that if A, B, C are sets and $A \subseteq B$, then $A^C \subseteq B^C$.

Solution: Let $f \in A^C$. Then, $f : C \to A$. Since $A \subseteq B$, $f(C) \subseteq A \subseteq B$, and $f : C \to B$. Thus, $f \in B^C$.

(c) Show that the sets $\{1,2\}^{\{1,2\}}$ and $\{1,2\}^{\{1,2,3\}}$ are disjoint.

Solution: Suppose the sets $\{1,2\}^{\{1,2\}}$ and $\{1,2\}^{\{1,2,3\}}$ are not disjoint. Then, there exists an f where $f \in \{1,2\}^{\{1,2\}}$ and $f \in \{1,2\}^{\{1,2,3\}}$. Since $f \in \{1,2\}^{\{1,2\}}$, the domain of f is the set $\{1,2\}$. But, since $f \in \{1,2\}^{\{1,2,3\}}$, the domain of f is the set $\{1,2,3\}$. But f cannot have two different domains. Contradiction $\lightning$. Thus, f does not exist and $\{1,2\}^{\{1,2\}}$ and $\{1,2\}^{\{1,2,3\}}$ are disjoint.

(d) Generalize part (c) by showing that if $B \neq C$ then $A^B \cap A^C = \varnothing$.

Solution: Suppose that $B \neq C$ and $A^B \cap A^C \neq \varnothing$. Then, there exists a function f where $f \in A^C$ and $f \in A^B$. Since $f \in A^C$, the domain of f is the set C. Similarly, since $f \in A^B$, the domain of f is B. So, the domain of f is both C and B simultaneously, but by assumption, $C \neq B$. Contradiction $\lightning$. No function can have two different domains. Thus, no such f exists, and $A^B \cap A^C = \varnothing$.

(e) Show that if A is a nonempty set, then $\varnothing^A = \varnothing$.

Solution: $\varnothing^A = \{$all functions f from A to $\varnothing\}$ and no such function exists. Thus, $\varnothing^A = \varnothing$.

(f) Show that if B is any set then $B^{\varnothing} = \{\varnothing\}$.

Solution: $B^{\varnothing} = \{\text{all functions } f \text{ s.t. } f : \varnothing \to B\}$. If f is a function in that set, then $f \subseteq \varnothing \times B$. The only subset that f could be is the $\varnothing$. Thus, $\{\varnothing\} = B^{\varnothing}$.

4. In each case, state without proof whether the given function is injective, surjective, and/or bijective.

(a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^3$.

Solution: f is injective, but not surjective since $\nexists x \in \mathbb{N}$ such that $f(x) = 5$.

(b) $g : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $g = \{(1, 5), (2, -2), (3, 6)\}$.

Solution: g is bijective.

(c) $h : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $h = \{(3, 5), (2, -2), (1, 5)\}$.

Solution: h neither injective nor surjective.

5. Let S and T by sets with three elements and two elements, respectively. In each case, state the answer and justify briefly.

(a) How many functions are there from S to T?

Solution: There are 8 functions from S to T. To see this, note that each of the three elements in S has two "choices" in T to be mapped to. This gives $2 \times 2 \times 2 = 8$ total possible "choices" of functions from S to T.

(b) How many injections are there from S to T?

Solution: There are no injections from S to T. To justify this, let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$. If we assume that f is an injection from S to T, then $f(s_1)$ and $f(s_2)$ must be distinct elements in T, therefore $t_1, t_2 \in \{f(s_1), f(s_2)\}$. But since $f(s_3)$ must either be t_1 or t_2, then $f(s_3) \in \{f(s_1), f(s_2)\}$, contradicting the assumption that f was a bijection (since this means that $f(s_3) = f(s_1)$ or $f(s_3) = f(s_2)$). Therefore no bijection exists.

(c) How many injections are there from T to S?

Solution: There are 6 injections from S to T. One way to see this is that the first element in T has 3 "choices" of elements to choose from; once this choice is made, the second element only has 2 remaining "choices" from which to choose, and therefore we have $3 \times 2 = 6$ injections. Alternatively, we can list them:
Let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$.

$$f_1 = \{(t_1, s_1), (t_2, s_2)\}$$

$$
\begin{aligned}
f_2 &= \{(t_1, s_1), (t_2, s_3)\} \\
f_3 &= \{(t_1, s_2), (t_2, s_1)\} \\
f_4 &= \{(t_1, s_2), (t_2, s_3)\} \\
f_5 &= \{(t_1, s_3), (t_2, s_1)\} \\
f_6 &= \{(t_1, s_3), (t_2, s_2)\}
\end{aligned}
$$

(d) How many surjections are there from S to T?

Solution: There are 6 surjections from S to T. To see this, suppose that s_1 gets mapped to t_1. Then one or both of s_2 and s_3 needs to be mapped to t_2 to ensure we have a subjection. There are 3 ways for this to happen:

$$
\begin{aligned}
&s_2 \mapsto t_1 \text{ and } s_3 \mapsto t_2 \\
&s_2 \mapsto t_2 \text{ and } s_3 \mapsto t_1 \\
&s_2 \mapsto t_2 \text{ and } s_3 \mapsto t_2
\end{aligned}
$$

By the same reasoning if s_1 gets mapped to t_2 then there are three possibilities. So, since s_1 goes to either t_1 or t_2 we have $3 + 3 = 6$ total surjections.

(e) How many surjections are there from T to S?

Solution: There are no surjections from T to S. Since S has 3 elements, the only way for a function $f : \{t_1, t_2\} \to \{s_1, s_2, s_3\}$ to be a surjection is if $\{f(t_1), f(t_2)\}$ has 3 elements, which is clearly impossible.

(f) Guess the answers to (a) and (b) if S has m elements and T has n elements.

Solution: For a: each of the m elements of S has n "choices" to be mapped to in T, therefore a good (and correct!) guess is that there are

$$
\underbrace{n \times n \times \cdots \times n}_{m \text{ factors}} = n^m
$$

functions $f : S \to T$.

For b: *Case* $n < m$: then there will be no injections by the same (pigeon hole principle) argument used in b.

Case $n \geqslant m$: Let's reason as in part c. The first element in S will have n "choices" in T, the second will then have the remaining $n - 1$ elements in T to "choose" from, the third will then have the remaining $n - 2$ elements in T to "choose" from, ..., the mth element will have the remaining $n - m + 1$ elements to choose from. So, a good (and correct!) guess is

$$
n \times (n-1) \times (n-2) \times \cdots \times (n - m + 1) = \frac{n!}{(n-m)!}
$$

injections.

6. * A function $f : A \to B$ is a subset of $A \times B$ and hence a subset of $(A \cup B) \times (A \cup B)$. By our definition of relation, f is a relation on $A \cup B$. Therefore, f has an inverse relation

$$f^{-1} = \{(b, a) \mid (a, b) \in f\}.$$

   (a) Write the inverses of

      i. $g : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $g = \{(1, 5), (2, -2), (3, 6)\}$.

      Solution:
      $$g^{-1} = \{(5, 1), (-2, 2), (6, 3)\}$$

      ii. $h : \{1, 2, 3\} \to \{-2, 5, 6\}$ given by $h = \{(3, 5), (2, -2), (1, 5)\}$.

      Solution: h is not injective since $f(3) = 5 = f(1)$ and $3 \neq 1$. Hence, the inverse function h^{-1} does not exist.

   (b) Show that if f is not injective, then f^{-1} is not a function.

   Solution: Suppose f is not 1-1. Then, for some $a_1, a_2 \in A$, $f(a_1) = b = f(a_2)$ and $a_1 \neq a_2$. So, since $(a_1, b), (a_2, b) \in f$, $(b, a_1), (b, a_2) \in f^{-1}$. However, the element b in the domain of f^{-1} does not have a unique output. Thus, f^{-1} is not a function.

   (c) Show that if f is injective but not surjective, then f^{-1} is a function whose domain is a proper subset of B.

   Solution: Suppose f is 1-1, but not onto. Since $f^{-1} = \{(f(a), a) \mid (a, f(a)) \in f\}$, f^{-1} is a function since every $f(a) \in f(A)$ has a unique $a \in A$ (a result of f being 1-1). In addition, the domain of f^{-1} is $f(A) \subset B$ (a result of f not being onto).

7. Let a and b be real numbers. Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = ax + b$.

   (a) Under what conditions on a and b is f a bijection from $\mathbb{R}$ to $\mathbb{R}$?

   Solution: f is bijective if and only if $a \neq 0$. To see this, if $a = 0$, then for all $x \in \mathbb{R}$, $f(x) = b$ and is therefore not injective. If $a \neq 0$, then confirm that $g(x) = \frac{x-b}{a}$ is such that $f \circ g(x) = x$ and $g \circ f(x) = x$ for every $x \in \mathbb{R}$. In other symbols, $g(x) = f^{-1}(x)$ is a function for all $x \in \mathbb{R}$, which by problem 6 implies that f is a bijection.

   (b) Under what conditions on a and b is the restriction $f\restriction_{\mathbb{Z}}$ a bijection from $\mathbb{Z}$ to $\mathbb{Z}$?

   Solution: f is bijective if and only if $a = \pm 1$ and $b \in \mathbb{Z}$. To see this, note that $f(0) = b$ and $f(1) = a + b$. This shows that $b \in f(\mathbb{Z})$ and $a + b \in f(\mathbb{Z})$. Therefore, if $f(\mathbb{Z}) \subseteq \mathbb{Z}$ then $b \in Z$ and $a + b \in \mathbb{Z}$, but this also implies that $(a + b) - b = a \in \mathbb{Z}$. On the other hand, if $a, b \in \mathbb{Z}$, then for any $x \in \mathbb{Z}$, $f(x) = ax + b \in Z$. What we have so far shown is that $f(\mathbb{Z}) \subseteq \mathbb{Z}$ if and only if $a, b \in \mathbb{Z}$. Now, suppose that $a \neq \pm 1$ then there is no element $x \in \mathbb{Z}$ such that $f(x) = b + 1$ (nor $b - 1$ for that matter). However, if $a = \pm 1$, then for every $x \in \mathbb{Z}$, $\frac{x-b}{a} \in \mathbb{Z}$, and therefore $g(x) = \frac{x-b}{a}$ will be a function from $\mathbb{Z}$ to $\mathbb{Z}$, and as in the previous part, $g = f^{-1}$, proving that f is bijective.

(c) Under what conditions on a and b is the restriction $f\restriction_{\mathbb{N}}$ a bijection from $\mathbb{N}$ to $\mathbb{N}$?

Solution: For f to be bijective, $a = 1$ and $b = 0$. It is clear that if $a = 1$ and $b = 0$ then $f(x) = x$ will give us a bijection. We now show that it is the only choice. From the previous part, since we want $f(\mathbb{N}) \subseteq \mathbb{N}$, we will (at very least) need $a, b \in \mathbb{Z}$. If $a < 0$ then for large enough $x \in \mathbb{N}$, $ax + b < 0$, meaning that $f(x) \notin \mathbb{N}$. Therefore, $a \geqslant 0$. If $a = 0$, then $f(x) = b$ will not be injective. We conclude here that $a > 0$; that is, $a \in \mathbb{N}$. Note that $f(1) = a + b \in \mathbb{N}$, therefore $a + b + 1 \in \mathbb{N}$; however, if $a > 1$, then there is no $x \in \mathbb{N}$ such that $f(x) = a + 1 + b$ (indeed, if $x \in \mathbb{N}$ and $x \neq 1$ then $x = 1 + k$ for some $k \in \mathbb{N}$; hence $f(x) = a + ka + b > a + 1 + b$ since $ka \geqslant a > 1$), so for $f\restriction_{\mathbb{N}}$ to have any chance of being a bijection, $a = 1$. From here we can assume $a = 1$. If $x, y \in \mathbb{N}$ and $x < y$, then $f(x) = x + b < y + b = f(y)$. Therefore, $f(1)$ will be the smallest element in $f(\mathbb{N})$, and hence, if f is going to be a bijection, $f(1) = 1$, but this forces $b = 0$.

8. Define the 1-tuple $(a_1) = \{a_1\}$. Define the 2-tuple (a_1, a_2) to be the ordered pair (a_1, a_2). In general, if the k-tuple $(a_1, a_2, \ldots, a_k)$ has been defined, define the $(k+1)$-tuple $(a_1, a_2, \ldots, a_k, a_{k+1})$ to be the ordered pair
$$\Big((a_1, \ldots, a_k), a_{k+1}\Big).$$
Prove by induction on n that this recursive definition of n-tuple retains the following essential property:
$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \iff a_i = b_i \text{ for each } i,\ 1 \leqslant i \leqslant n\,.$$

Solution: Proof via induction on n:

$P(1)$: Suppose $n = 1$. Then, $(a_1) = (b_1)$ if and only if $a_1 = b_1$ is true. Since our reasoning involves a recursive formula, we will check $P(2)$.

$P(2)$: Suppose $n = 2$. $(a_1, a_2) = (b_1, b_2)$ is a sequence of length 2. By definition, this means there is a function $f : \mathbb{N}_2 \to A$ such that $f(1) = a_1$ and $f(2) = a_2$, and there is a function $g : \mathbb{N}_2 \to B$ such that $g(1) = b_1$ and $g(2) = b_2$. These sequences are equal if and only the functions are equal at every i^{th} coordinate. Thus, $(a_1, a_2) = (b_1, b_2)$ iff $f(i) = g(i)$ for all i, $1 \leqslant i \leqslant 2$. So, $(a_1, a_2) = (b_1, b_2)$ iff $a_i = b_i$ for all i, $1 \leqslant i \leqslant 2$.

Assume $P(k)$ is true. That is, $(a_1, \ldots, a_k) = (b_1, \ldots, b_k)$ iff $a_i = b_i$ for all $1 \leqslant i \leqslant k$.

$$\begin{aligned}
&(a_1, \ldots, a_k, a_{k+1}) = (b_1, \ldots, b_k, b_{k+1}) \\
&\iff \big((a_1, \ldots, a_k), a_{k+1}\big) = \big((b_1, \ldots, b_k), b_{k+1}\big) \\
&\iff (a_1, \ldots, a_k) = (b_1, \ldots, b_k) \text{ and } a_{k+1} = b_{k+1}
\end{aligned}$$

By our hypothesis, $a_i = b_i$ for all i, $1 \leqslant i \leqslant k$. Thus, $(a_1, \ldots, a_k, a_{k+1}) = (b_1, \ldots, b_k, b_{k+1})$ if and only if $a_i = b_i$ for all i, $1 \leqslant i \leqslant k + 1$, and $P(k + 1)$ is true.

By the Principle of Mathematical Induction, the recursive definition of n-tuple retains the essential property:
$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \iff a_i = b_i \text{ for each } i,\ 1 \leqslant i \leqslant n\,.$$

9. Information transmitted electronically is usually encoded in the form of n-tuples of 0s and 1s, often called **n-bit strings**. For brevity, commas and parentheses are omitted; thus, 01101011 is an 8-bit string. A **code of length** n is a set of such strings, and the members of code are called codewords. To aid in the detection and correction of errors, it is useful to have different codewords differ in more that one coordinate. Define the **distance** between two n-bit strings to be the number of coordinates in which they differ. More formally, let $\mathbb{Z}_2^n$ denote the set of n-bit strings, and define a function $d : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}$ by

$$d(a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_n) = \sum_{i=1}^{n} |a_i - b_i|$$

where $a_i, b_i \in \{0, 1\}$ for each i. For example, $d(1101, 1001) = 1$. Prove that the function d has the following properties:

(a) $d(s_1, s_2) \geqslant 0$

Solution: Let $s_1 = a_1 a_2 a_3 \ldots a_n$ and $s_2 = b_1 b_2 b_3 \ldots b_n$. Then,

$$\begin{aligned} d(s_1, s_2) &= d(a_1 a_2 a_3 \ldots a_n, b_1 b_2 b_3 \ldots b_n) \\ &= \sum_{i=1}^{n} |a_i - b_i| \\ &= |a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n| \\ &\geqslant 0 \quad \text{(since absolute value of any number is greater than 0} \end{aligned}$$

(b) $d(s_1, s_2) = 0 \iff s_1 = s_2$

Solution: Let $s_1 = a_1 a_2 a_3 \ldots a_n$ and $s_2 = b_1 b_2 b_3 \ldots b_n$. Then,

$$\begin{aligned} d(s_1, s_2) &= d(a_1 a_2 a_3 \ldots a_n, b_1 b_2 b_3 \ldots b_n) \\ &= \sum_{i=1}^{n} |a_i - b_i| \\ &= |a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n| \end{aligned}$$

Thus,

$$\begin{aligned} d(s_1, s_2) = 0 &\iff |a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n| = 0 \\ &\iff |a_i - b_i| = 0 \quad \text{for all } i,\ 1 \leqslant i \leqslant n \\ &\iff a_i = b_i \quad \text{for all } i,\ 1 \leqslant i \leqslant n \\ &\iff s_1 = s_2 \end{aligned}$$

(c) $d(s_1, s_2) = d(s_2, s_1)$

Solution: Let $s_1 = a_1a_2a_3 \ldots a_n$ and $s_2 = b_1b_2b_3 \ldots b_n$. Then,

$$\begin{aligned} d(s_1, s_2) &= d(a_1a_2a_3 \ldots a_n, b_1b_2b_3 \ldots b_n) \\ &= \sum_{i=1}^{n} |a_i - b_i| \\ &= |a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n| \\ &= |b_1 - a_1| + |b_2 - a_2| + \ldots + |b_n - a_n| \\ &= \sum_{i=1}^{n} |b_i - a_i| \\ &= d(b_1b_2b_3 \ldots b_n, a_1a_2a_3 \ldots a_n) \\ &= d(s_2, s_1) \end{aligned}$$

(d) For all $s_1, s_2, s_3 \in \mathbb{Z}_2^n$,

$$d(s_1, s_3) \leqslant d(s_1, s_2) + d(s_2, s_3) \qquad \text{(the \textbf{triangle inequality})}$$

Suggestion: Use induction on n, starting with $n = 1$.

Solution: Proof via induction on n.
Let $s_1 = a_1a_2 \ldots a_n$, $s_2 = b_1b_2b_3 \ldots b_n$, and $s_3 = c_1c_2c_3 \ldots c_n$.
$P(1)$: Suppose $n = 1$. So, $s_1 = a_1$, $s_2 = b_2$, and $s_3 = c_3$, and

$$\begin{aligned} d(s_1, s_3) &= |a_1 - c_1| \\ &= |a_1 - b_1 + b_1 - c_1| \\ &= |(a_1 - b_1) + (b_1 - c_1)| \\ &\leqslant |a_1 - b_1| + |b_1 + c_1| \\ &= d(s_1, s_2) + d(s_2, s_3) \end{aligned}$$

And, $P(1)$ is true.
Now, assume $P(k)$ is true. We will show that $P(k + 1)$ is true.

Let $s_1 = a_1a_2...a_ka_{k+1}$, $s_2 = b_1b_2...b_kb_{k+1}$, and $s_3 = c_1c_2...c_kc_{k+1}$. Also, let $\tilde{s}_1 = a_1a_2...a_k$, $\tilde{s}_2 = b_1b_2...b_k$, and $\tilde{s}_3 = c_1c_2...c_k$. By the inductive hypothesis $d(\tilde{s}_1, \tilde{s}_3) \leqslant$

$d(\tilde{s}_1, \tilde{s}_2) + d(\tilde{s}_2, \tilde{s}_3)$. From here

$$\begin{aligned}
d(s_1, s_3) &= \sum_{i=1}^{k+1} |a_i - c_i| \\
&= \sum_{i=1}^{k} |a_i - c_i| + |a_{k+1} - c_{k+1}| \\
&= d(\tilde{s}_1, \tilde{s}_3) + |a_{k+1} - c_{k+1}| \\
&\leqslant d(\tilde{s}_1, \tilde{s}_2) + d(\tilde{s}_2, \tilde{s}_3) + |a_{k+1} - c_{k+1}| \\
&= \sum_{i=1}^{k} (|a_i - b_i| + |b_i - c_i|) + |(a_{k+1} - b_{k+1}) + (b_{k+1} - c_{k+1})| \\
&\leqslant \sum_{i=1}^{k} (|a_i - b_i| + |b_i - c_i|) + (|a_{k+1} - b_{k+1}| + |b_{k+1} - c_{k+1}|) \\
&= \sum_{i=1}^{k+1} (|a_i - b_i| + |b_i - c_i|) \\
&= d(s_1, s_2) + d(s_2, s_3)
\end{aligned}$$

Thus, $P(k+1)$ is true, and by the Principle of Mathematical Induction,

$$d(s_1, s_3) \leqslant d(s_1, s_2) + d(s_2, s_3)$$

is true for all n.

10. * Suppose $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are surjections. Prove that there is a surjection $h : \mathbb{N} \to A \cup B$. [Suggestion: Consider the list $f(1), g(1), f(2), g(2), \ldots$]

Solution: Let h be the function from $\mathbb{N}$ to $A \cup B$ defined as follows:

$$h(i) = \begin{cases} f\left(\dfrac{i}{2}\right) & \text{if } i \text{ is even} \\ g\left(\dfrac{i+1}{2}\right) & \text{if } i \text{ is odd .} \end{cases}$$

Now, we will show that h is surjective. Let $c \in A \cup B$. Then, $c \in A$ or $c \in B$. If $c \in A$, then since f is surjective, there exists an $a \in A$ such that $f(a) = c$. Then, $h(2a) = f(a) = c$. Similarly, if $c \in B$ and g is onto, then there exists $b \in B$ such that $g(b) = c$. Then, $h(2b - 1) = g(b) = c$. Thus, h is onto.