Homework 9 (Due Tues, Apr 8)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. * Let g, r, s be functions, and let f be a bijective function. Prove the following:
   (a) $f \circ g = f \circ h \implies g = h$ (left cancellation),
   (b) $r \circ f = s \circ f \implies r = s$ (right cancellation).

2. Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
   (Hint: Use the following theorem appropriately that we covered in class: If $f : A \to B$ and $g : B \to A$, then the following are equivalent:
   (a) f is a bijection and $g = f^{-1}$
   (b) $g \circ f = i_A$ and $f \circ g = i_B$.

3. Suppose $f : A \to B$ and $g : B \to C$ are functions.
   (a) Show that if $g \circ f$ is injection, then f is injective.
   (b) Show that is $g \circ f$ is surjective, then g is surjective.

4. Suppose that A and B are non-empty sets. Let $f : A \to B$. Given any subset $V \subseteq B$, the *inverse image* of V with respect to f is the set

$$f^{-1}(V) = \{x \in A \mid f(x) \in V\} \subseteq A.$$

   The notation f^{-1} in this context does *not* imply that f is invertible, but it is a convenient and widely used notation for the inverse image of V; this definition makes sense for any function.
   (a) * Let $V \subseteq B$ and $U = f^{-1}(V)$. Prove that $f(U) = V$ if and only if $V \subseteq f(A)$.
   (b) If $V_1, V_2 \subseteq B$, prove that

$$f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2).$$

   (c) If $V_1, V_2 \subseteq B$, prove that

$$f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2).$$

   (d) If $V \subseteq B$, prove that

$$f^{-1}(V^c) = f^{-1}(V)^c.$$

   Note that $V^c = B - V$ (i.e. V^c is the compliment of V in B) and $f^{-1}(V)^c = A - f^{-1}(V)$ (i.e. $f^{-1}(V)^c$ is the complement of $f^{-1}(V)$ in A).

(e) If $f : A \to B$ is a bijection and $V \subseteq B$, prove that

$$f^{-1}(V) = \{f^{-1}(y) \mid y \in V\}.$$

Here, on the left-hand side, $f^{-1}(V)$ means the inverse image of V with respect to f; whereas, on the right-hand side $f^{-1}(y)$ means the inverse of f (which exists since we've assumed f was bijective).