Homework 9 (Due Tues, Apr 8)

Math 2710 – Spring 2014
Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. * Let g, h, r, s be functions, and let f be a bijective function. Prove the following:

   (a) $f \circ g = f \circ h \implies g = h$ (left cancellation),

   Solution: Suppose $f : B \to C$ is a bijective function and let $g : A \to B$ and $h : A \to B$ are functions. Since f is bijective, f^{-1} exists and $f^{-1} \circ f = i_B$. Assume $f \circ g = f \circ h$. Then,

   $$\begin{aligned} f \circ g = f \circ h &\implies f^{-1} \circ f \circ g = f^{-1} \circ f \circ h \\ &\implies i_B \circ g = i_B \circ h \\ &\implies g = h \end{aligned}$$

   (b) $r \circ f = s \circ f \implies r = s$ (right cancellation).

   Solution: Suppose $f : A \to B$ is a bijective function and let $r : B \to C$ and $s : B \to C$ are functions. Assume $r \circ f = s \circ f$. Since f is bijective, f^{-1} exists, and for every $a \in A$, $a = f^{-1}(b)$ for some $b \in B$. In addition, $f \circ f^{-1} = i_B$. Then,

   $$\begin{aligned} (r \circ f)(a) = (s \circ f)(a) &\implies r\big(f(f^{-1}(b))\big) = s\big(f(f^{-1}(b))\big) \\ &\implies r\big(i_B(b)\big) = s\big(i_B(b)\big) \\ &\implies r(b) = s(b) \end{aligned}$$

   Thus, $r = s$.

2. Suppose that A and B are non-empty sets. Let $f : A \to B$. Given any subset $V \subseteq B$, the *inverse image* of V with respect to f is the set

   $$f^{-1}(V) = \{x \in A \mid f(x) \in V\} \subseteq A.$$

   The notation f^{-1} in this context does *not* imply that f is invertible, but it is a convenient and widely used notation for the inverse image of V; this definition makes sense for any function.

   (a) * Let $V \subseteq B$ and $U = f^{-1}(V)$. Prove that $f(U) = V$ if and only if $V \subseteq f(A)$.

Solution: ($\Longrightarrow$) Notice that if $V = \varnothing$, $V \subseteq f(A)$, and we're done. Now, assume $f(U) = V$, and let $v \in V$. Then, $v = f(u)$ for some $u \in U$. Since $U = f^{-1}(V) \subseteq A$, $v = f(u)$ for some $u \in A$. Hence, $v \in f(A)$. Thus, $V \subseteq f(A)$.

($\Longleftarrow$) Assume $V \subseteq f(A)$. Let $y \in f(U)$. Then,

$$\begin{aligned} y \in f(U) &\iff y = f(u) \text{ for some } u \in U \\ &\iff y = f(u),\ u \in f^{-1}(V) \\ &\iff y = f(u) \text{ for } u \in A \text{ and } f(u) \in V \\ &\iff y \in V. \end{aligned}$$

3. * (pg. 154 # 37) If $f : X \to Y$ is a bijective function, prove that its inverse is unique.

Solution: Since f is bijective, we know that an inverse function exists. Suppose that the inverse function is not unique. Then, we at least two different functions that act as an inverse of f. Let us call those functions $g : Y \to X$ and $h : Y \to X$. Since g is an inverse of f, for every $x \in X$, $(g \circ f)(x) = x$ Similarly, since h is an inverse of f, for every $x \in X$, $(h \circ g)(x) = x$. Thus, $g \circ f = h \circ f$ for all $x \in X$. Since f is bijective, by Homework 9, Question 1(b), we have right cancellation of f, and $g = h$. Thus, the inverse of f is unique.