Homework 9 (Due Tues, Apr 8)

Math 2710 – Spring 2014

Professor Hohn

Using the proof techniques we have learned in class, prove each statement.

1. * Let g, r, s be functions, and let f be a bijective function. Prove the following:
   (a) $f \circ g = f \circ h \implies g = h$ (left cancellation),
   (b) $r \circ f = s \circ f \implies r = s$ (right cancellation).

2. Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

   Hint: Use the following theorem appropriately that we covered in class: If $f : A \to B$ and $g : B \to A$, then the following are equivalent:
   (a) f is a bijection and $g = f^{-1}$
   (b) $g \circ f = i_A$ and $f \circ g = i_B$.

Solution: By the theorem above, if $(f^{-1} \circ g^{-1}) \circ (g \circ f) = i_A$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_A$, then $g \circ f$ is a bijection and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. So, we will show that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = i_A$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_A$.

$$\begin{aligned}(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ g^{-1} \circ g \circ f \\ &= f^{-1} \circ i_B \circ f \qquad \text{since } g \text{ is bijective, see theorem above} \\ &= f^{-1} \circ f \\ &= i_A \qquad \text{since } f \text{ is bijective, see theorem above.}\end{aligned}$$

Similarly,

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ f \circ f^{-1} \circ g \\ &= g^{-1} \circ i_B \circ g \qquad \text{since } f \text{ is bijective, see theorem above} \\ &= g^{-1} \circ g \\ &= i_A \qquad \text{since } g \text{ is bijective, see theorem above.}\end{aligned}$$

Hence, since $(f^{-1} \circ g^{-1}) \circ (g \circ f) = i_A$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_A$, by the theorem above, $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

3. Suppose $f : A \to B$ and $g : B \to C$ are functions.

(a) Show that if $g \circ f$ is injection, then f is injective.

Solution: Suppose $g \circ f$ is 1-1. That is, for $a_1, a_2 \in A$, if $g(f(a_1)) = g(f(a_2))$, then $a_1 = a_2$. Now, suppose $a_1, a_2 \in A$ and that $f(a_1) = f(a_2)$. Then, by composing on the left with g, $g(f(a_1)) = g(f(a_2))$. Since g is 1-1, $a_1 = a_2$. Hence, f is injective.

(b) Show that if $g \circ f$ is surjective, then g is surjective.

Solution: Suppose $g \circ f$ is onto. That is, if $c \in C$, then there exists an $a \in A$ such that $g(f(a)) = c$. We want to show that g is onto. Let $c \in C$. We want to find an element in B such that $g(b) = c$. We know that $f(a) \in B$ and $g(f(a)) = c$. So, the element in the domain of g that maps to c is $f(a) \in B$. Thus, g is onto.

4. Suppose that A and B are non-empty sets. Let $f : A \to B$. Given any subset $V \subseteq B$, the *inverse image* of V with respect to f is the set

$$f^{-1}(V) = \{x \in A \mid f(x) \in V\} \subseteq A.$$

The notation f^{-1} in this context does *not* imply that f is invertible, but it is a convenient and widely used notation for the inverse image of V; this definition makes sense for any function.

(a) * Let $V \subseteq B$ and $U = f^{-1}(V)$. Prove that $f(U) = V$ if and only if $V \subseteq f(A)$.

Solution: ($\Longrightarrow$) Notice that if $V = \varnothing$, $V \subseteq f(A)$, and we're done. Now, assume $f(U) = V$, and let $v \in V$. Then, $v = f(u)$ for some $u \in U$. Since $U = f^{-1}(V) \subseteq A$, $v = f(u)$ for some $u \in A$. Hence, $v \in f(A)$. Thus, $V \subseteq f(A)$.

($\Longleftarrow$) Assume $V \subseteq f(A)$. Let $y \in f(U)$. Then,

$$\begin{aligned} y \in f(U) &\iff y = f(u) \text{ for some } u \in U \\ &\iff y = f(u),\ u \in f^{-1}(V) \\ &\iff y = f(u) \text{ for } u \in A \text{ and } f(u) \in V \\ &\iff y \in V. \end{aligned}$$

(b) If $V_1, V_2 \subseteq B$, prove that

$$f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2).$$

Solution: Let $V_1, V_2 \subseteq B$. Then,

$$\begin{aligned} x \in f^{-1}(V_1 \cup V_2) &\iff x \in A \text{ and } f(x) \in V_1 \cup V_2 \\ &\iff x \in A \text{ and } (f(x) \in V_1 \text{ or } f(x) \in V_2) \\ &\iff (x \in A \text{ and } f(x) \in V_1) \text{ or } (x \in A \text{ and } f(x) \in V_2) \\ &\iff (x \in f^{-1}(V_1)) \text{ or } (x \in f^{-1}(V_2)) \\ &\iff x \in f^{-1}(V_1) \cup f^{-1}(V_2). \end{aligned}$$

(c) If $V_1, V_2 \subseteq B$, prove that

$$f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2).$$

Solution: Let $V_1, V_2 \subseteq B$. Then,

$$\begin{aligned} x \in f^{-1}(V_1 \cap V_2) &\iff x \in A \text{ and } f(x) \in V_1 \cap V_2 \\ &\iff x \in A \text{ and } (f(x) \in V_1 \text{ and } f(x) \in V_2) \\ &\iff (x \in A \text{ and } f(x) \in V_1) \text{ and } (x \in A \text{ and } f(x) \in V_2) \\ &\iff (x \in f^{-1}(V_1)) \text{ and } (x \in f^{-1}(V_2)) \\ &\iff x \in f^{-1}(V_1) \cap f^{-1}(V_2). \end{aligned}$$

(d) If $V \subseteq B$, prove that

$$f^{-1}(V^c) = f^{-1}(V)^c.$$

Note that $V^c = B - V$ (i.e. V^c is the compliment of V in B) and $f^{-1}(V)^c = A - f^{-1}(V)$ (i.e. $f^{-1}(V)^c$ is the complement of $f^{-1}(V)$ in A).

Solution: Let $V \subseteq B$. Then,

$$\begin{aligned} x \in f^{-1}(V^c) &\iff x \in A \text{ and } f(x) \in V^c \\ &\iff x \in A \text{ and } f(x) \in B - V \\ &\iff x \in A \text{ and } (f(x) \in B \text{ and } f(x) \notin V) \\ &\iff (x \in A \text{ and } f(x) \notin V) \text{ and } f(x) \in B \\ &\iff x \in f^{-1}(V)^c. \end{aligned}$$

(e) If $f : A \to B$ is a bijection and $V \subseteq B$, prove that

$$f^{-1}(V) = \{f^{-1}(y) \mid y \in V\}.$$

Here, on the left-hand side, $f^{-1}(V)$ means the inverse image of V with respect to f; whereas, on the right-hand side $f^{-1}(y)$ means the inverse of f (which exists since we've assumed f was bijective).

Solution: Suppose f is bijective and $V \subseteq B$. Thus, f has an inverse function f^{-1} and we can say that an element $x \in A$ can be written as $x = f^{-1}(y)$ for some $y \in B$. Notice that since f is bijective, we also have $f(A) = B$.

$$\begin{aligned} x \in f^{-1}(V) &\iff x \in A \text{ and } f(x) \in V \\ &\iff f^{-1}(y) \in A \text{ and } f(f^{-1}(y)) \in V \\ &\iff f^{-1}(y) \in A \text{ and } y \in V. \end{aligned}$$