PRACTICE PROBLEMS FOR EXAM 1

Math 3160Q – Fall 2015
Professor Hohn

Below is a list of practice questions for Exam 1. Any quiz, homework, or example problem has a chance of being on the exam. For more practice, I suggest you work through the review questions at the end of each chapter as well.

1. As Mario is walking, he stumbles upon three, large colored sewer pipes: Pipe I, Pipe II, and Pipe III. After some inspection, he finds that inside Pipe I, there are 10 red coins and 7 yellow coins. In Pipe II, he finds 8 red and 8 yellow coins. In Pipe III, he finds 2 red and 10 yellow coins. Luigi, who enjoys collecting coins, sees Mario and jaunts over to the three pipes. Mario (who knows Luigi well) believes that Luigi would jump into Pipe I with probability 0.3, the Pipe II with probability 0.2, and Pipe III with probability 0.5.

   (a) While Mario is checking out the surrounding area for possible threats (Koopas...ahhh!!), Luigi jumps into one of the pipes and returns with a coin. If Luigi is equally likely to grab any coin, what is the probability that Luigi picked a red coin?

   Solution: Let P_i = Luigi goes down Pipe i and let R = Luigi picks a red coin. From the question, we know $P(P_1) = 0.3$, $P(P_2) = 0.2$, and $P(P_3) = 0.5$. We want to know $P(R)$.

   $$\begin{aligned} P(R) &= P(R \cap P_1) + P(R \cap P_2) + P(R \cap P_3) \\ &= P(P_1)P(R \mid P_1) + P(P_2)P(R \mid P_2) + P(P_3)P(R \mid P_3) \end{aligned}$$

   We are told that inside Pipe I there are 10 red coins and 7 yellow coins. Then,

   $$P(R \mid P_1) = \frac{10}{10+7} = \frac{10}{17}.$$

   Similarly, we are told that inside Pipe II there are 8 red coins and 8 yellow coins. Then,

   $$P(R \mid P_2) = \frac{8}{8+8} = \frac{8}{16} = \frac{1}{2}.$$

   Inside Pipe III there are 2 red coins and 10 yellow coins. So,

   $$P(R \mid P_3) = \frac{2}{2+10} = \frac{2}{12} = \frac{1}{6}.$$

   Now, we plug in our information into our equation for $P(R)$.

   $$\begin{aligned} P(R) &= P(R \cap P_1) + P(R \cap P_2) + P(R \cap P_3) \\ &= P(P_1)P(R \mid P_1) + P(P_2)P(R \mid P_2) + P(P_3)P(R \mid P_3) \\ &= 0.3\left(\frac{10}{17}\right) + 0.2\left(\frac{1}{2}\right) + 0.5\left(\frac{1}{6}\right) \\ &= \frac{3}{17} + \frac{1}{10} + \frac{1}{12} \end{aligned}$$

(b) Given Luigi picked a red coin, what is the probability that the coin came from Pipe II?

Solution: We want to find $P(P_2 \mid R)$, so we use Bayes' formula to help us.

$$\begin{aligned} P(P_2 \mid R) &= \frac{P(P_2)P(R \mid P_2)}{P(R)} && \text{by Bayes' formula} \\ &= \frac{0.2\left(\frac{1}{2}\right)}{\frac{3}{17}+\frac{1}{10}+\frac{1}{12}} && P(R) \text{ from Part (a)} \end{aligned}$$

2. An urn initially contains 5 red marbles and 7 blue marbles. Bored out of your mind, you decide to play a "game" that goes as follows. During each round, you randomly pull a marble out of the urn. If the marble you chose was red, you return the marble back into the urn along with 1 more blue marble. If the marble you chose was blue, you put the marble back into the urn along with 2 more red marbles. What is the probability that on the first round you drew a red marble and on the third round you drew a blue marble?

Solution: Let R_i be the event that you drew a red marble on the ith round and B_i be the event that you drew a blue marble on the ith round ($i = 1, 2, 3$). We are looking for $P(R_1 \cap B_3)$. We divide this into the cases which consider what you drew on the second round,

$$P(R_1 \cap B_3) = P(R_1 \cap R_2 \cap B_3) + P(R_1 \cap B_2 \cap B_3).$$

Now, using the multiplication rule

$$\begin{aligned} P(R_1 \cap B_3) &= P(R_1)P(R_2|R_1)P(B_3|R_2 \cap R_1) + P(R_1)P(B_2|R_1)P(B_3|B_2 \cap R_1) \\ &= \frac{5}{7+5}\cdot\frac{5}{8+5}\cdot\frac{9}{9+5} + \frac{5}{7+5}\cdot\frac{8}{8+5}\cdot\frac{8}{8+7} \\ &= \frac{5}{12}\cdot\frac{5}{13}\cdot\frac{9}{14} + \frac{5}{12}\cdot\frac{8}{13}\cdot\frac{8}{15} \end{aligned}$$

3. Your professor hands you three coins. The first coin has a probability of .6 of landing heads; the second coin has a probability .3 of landing heads; the third coin has a probability .5 of landing heads (i.e., the third coin is fair). You are asked to conduct the following experiment: You flip the first coin and note the result (heads or tails). If the first flip resulted in heads, you then flip the second coin and note the result. If, on the other hand, the first flip resulted in a tails, you then flip the third coin and note the result.

(a) What is a reasonable sample space Ω for this experiment?

Solution: At the end of the experiment, we will have just flipped two coins. So, a reasonable sample space is

$$\Omega = \{HH, HT, TH, TT\}.$$

(b) Let X be (the random variable giving) the number of heads which occurred during the experiment. What is the state space S_X of X?

Solution: We will either flip $0, 1$, or 2 heads. Therefore $S_X = \{0, 1, 2\}$.

(c) If X is the same random variable as in the previous part and F_X is the cumulative distribution function (CDF) of X, what is $F_X(1.5)$?

Solution: $F_X(1.5) = P(X \leq 1.5)$. Since the event $\{X \leq 1.5\}$ is the same as $\{X = 0\} \cup \{X = 1\}$ which is the same as $\{TT\} \cup \{TH\} \cup \{HT\}$, we have

$$P(X \leq 1.5) = P(TT) + P(HT) + P(TH) = (.4)(.5) + (.6)(.7) + (.4)(.5) = .82.$$

4. A circuit has three nodes: A, B, and C. Each node is *independently* functional with probability p_A, p_B, and p_C respectively. The circuit works if either A is functional, or both B and C are functional. Otherwise, the circuit does not work. Find the probability that the circuit is working. (Your answer should be in terms of p_A, p_B, and p_C. This problem is not meant to be tricky!)

Solution: Let F be the event that the circuit is functional. Let A be the event that A is functional ($P(A) = p_A$), B be the event that B is functional ($P(B) = p_B$), and C be the event that C is functional ($P(C) = p_C$). Then, we want to find $P(F) = P(A \cup (B \cap C))$.

$$\begin{aligned} P(F) &= P(A \cup (B \cap C)) \\ &= P(A) + P(B \cap C) - P(A \cap (B \cap C)) && \text{inclusion/exclusion principle} \\ &= P(A) + P(B)P(C) - P(A)P(B)P(C) && \text{by independence} \\ &= p_A + p_B p_C - p_A p_B p_C \end{aligned}$$

5. After watching Guardians of the Galaxy, our class became extremely motivated to befriend a raccoon. In this class there are 20 males and 16 females.

(a) We are going to make two committees from the people in this class. The first committee will consist of 6 people, 3 males and 3 females, to search for a raccoon to befriend; the second committee will consist of 4 other people, 2 males and 2 females, to wait near a phone and medical supplies so that after the first committee actually finds a raccoon and realizes that they're vicious beasts, they can be bandaged up and an ambulance can be called. How many ways are there for us to create these committees?

Solution: We want to know how many ways to select 5 males and 5 females from the class to be on the committees, then find the number of ways these 5 males and 5 females can be put into two groups (one to befriend the raccoon and one to be ready with the medical supplies). First, we find out how many ways we can choose the ten students that will be on the committees.

$$\overbrace{\binom{20}{5}}^{\text{males}} \cdot \overbrace{\binom{16}{5}}^{\text{females}} = 67721472$$

Once these students are chosen, we have $\binom{5}{3}$ ways to select which 3 males will be on the first committee, and we have $\binom{5}{3}$ ways to select which 3 females will be on the first committee. The other 4 students will be on the second committee. This means that for the first committee, we have $\binom{5}{3}\binom{5}{3} = \binom{5}{3}^2 = 10^2 = 100$ ways to organize the 5 students into two committees. Hence, the number of ways to create the two committees is

$$\binom{20}{5}\binom{16}{5}\binom{5}{3}^2 = 15504 \cdot 4368 \cdot 100 = 6772147200$$

(b) After everyone survived the "raccoon incident" (and I lost my job), all 36 people in this class return to the classroom and shake everyone else's hands (no repeated handshakes) on a job well done. How many handshakes took place?

Solution: There are a couple ways to think about this problem. Here are two:

Way 1: Student 1 shakes 35 other hands, Student 2 shakes 34 other hands, Student 3 shakes 33 other hands, and so on. Then, we add up all of the handshakes.

$$35 + 34 + 33 + 32 + \ldots + 2 + 1 = 36 \cdot (35/2) = 630$$

Way 2: We can think of the handshake question as wanting to find out how many ways we can group two students together (who would then shake hands). That is, we want to know how many ways we can choose 2 from 36.

$$\binom{36}{2} = 630$$

6. Mr. Tom Riddle finally bought a computer, and he's looking to stay connected with his Death Eater friends on Instagram. He wants to create his new screen name by rearranging the letters

MRTOMRIDDLE

(a) How many different screen names (distinguishable letter arrangements) can he make such that M is the first and fifth letter in the screen name?

Solution: We have 11 total letters: 2 M's, 2 R's, 2 D's, 1 T, 1 O, 1 I, 1 L, and 1 E. If M is the first and fifth letter in the screen name, we have something that looks like

$$\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}$$

Once we fix where the two M's, the other 9 letters (represented above by $*$) can be rearranged in $\binom{9}{2, 2, 1, 1, 1, 1, 1} = \frac{9!}{2!2!}$ distinguishable ways by the word counting principle.

(b) Let M be the event that M is the first and fifth letter in the screen name, and let D be the event that the D's must be next to each other. What is the probability of $P(M \cap D)$?

Solution: First, we will determine the number of ways to get M as the first and fifth letter in the screen name and the D's next to each other (i.e. $\#M \cap D$).
If M is the first and fifth letter in the screen name, we have something that looks like

$$\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}.$$

Now, we want to see how the D's will be added.

$$7 \text{ positions for the 2 D's} = \begin{cases} \underline{M}\,\underline{D}\,\underline{D}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*} \\ \underline{M}\,\underline{*}\,\underline{D}\,\underline{D}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*} \\ \underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{M}\,\underline{D}\,\underline{D}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*} \\ \vdots \\ \underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{D}\,\underline{D} \end{cases}$$

For every position of D's, there are other 7 letters (represented above by $*$) that can be rearranged in $\binom{7}{2,1,1,1,1,1} = \frac{7!}{2!}$ distinguishable ways by the word counting principle. Thus, the number of ways we can have M's in the first and fifth position and the D's together are

$$7 \cdot \binom{7}{2,1,1,1,1,1} = \frac{7 \cdot 7!}{2!}$$

Since we are looking for $P(M \cap D)$, we need to find out the number of ways that we can arrange all of the letters. By the word counting principle, since there are 11 total letters: 2 M's, 2 R's, 2 D's, 1 T, 1 O, 1 I, 1 L, and 1 E, the total number of ways to arrange the letters is equal to

$$\binom{11}{2,2,2,1,1,1,1,1} = \frac{11!}{2!2!2!}$$

Thus,

$$P(M \cap D) = \frac{\frac{7 \cdot 7!}{2!}}{\frac{11!}{2!2!2!}}$$

7. You conduct an "experiment" where you continually flip a coin (keeping track of the outcome of each flip) until the either the first heads appears or the coin has been flipped 5 times, at which point you stop.

   (a) What is a reasonable *sample space* for this experiment?

   Solution: A reasonable sample space is

   $$\Omega = \{H, TH, TTH, TTTH, TTTTH, TTTTT\}$$

   where each element displays the outcomes of the flips until they stop. For example, TTH represents the first two flips landing tails and the third flip landing heads.

   (b) If E is the event that you flip the coin an odd number of times before stopping, describe the event E as a subset of the sample space. (For example: If a sample space is $\{1, 2, 3\}$ and F is the event that the outcome is even, then the description of F as a subset of $\{1, 2, 3\}$ is just $F = \{2\}$).

   Solution: If we collect together all the elements in Ω which correspond to flipping an odd number of times, we have

   $$E = \{H, TTH, TTTTH, TTTTT\}.$$

8. There are three urns: Urn A, Urn B, and Urn C. Urn A contains 4 red marbles and 6 blue marbles; Urn B contains 3 red marbles and 7 blue marbles; Urn C has 8 red marbles and 2 blue marbles. You are going to select a random marble from one of the urns; there is a 1/2 probability you will draw the marble from Urn A; there is a 3/10 probabilty you will draw a marble from Urn B; there is a 1/5 probability you will draw the marble from Urn C.

   (a) Let R be the event that you select a red marble. Find $P(R)$.

   Solution: Let E_A, E_B, and E_C be the events that you draw from Urn A, Urn B, and Urn C, respectively. Note that $\{E_A, E_B, E_C\}$ is a partition of all possible outcomes. Therefore,

$$P(R) = \overbrace{P(R \mid E_A)}^{4/10}\overbrace{P(E_A)}^{1/2} + \overbrace{P(R \mid E_B)}^{3/10}\overbrace{P(E_B)}^{3/10} + \overbrace{P(R \mid E_C)}^{8/10}\overbrace{P(E_C)}^{1/5}$$
$$= \frac{4}{20} + \frac{9}{100} + \frac{8}{50} = \frac{45}{100}$$

   (b) Given that the marble you draw is red, what is the probability that you drew the marble from Urn A?

Solution: We want $P(E_A \mid R)$. Using Bayes' formula,

$$P(E_A \mid R) = \frac{P(R \mid E_A)P(E_A)}{P(R)} = \frac{(4/10)(1/2)}{45/100} = \frac{4}{9}$$

(where we just plugged in the value for $P(R)$ that we found in part a.)

9. (a) How many distinguishable letter arrangements can be made from the word

BEYONCE

so that the E s are *not* next to each other?

Solution: *Method 1:* By the word counting principle, there are $\frac{7!}{2!}$ different distinguishable letter arrangements of BEYONCE. Grouping together both Es, we see there are 6! arrangements which result in the Es landing next to each other. So, the total number of distinguishable arrangements where the Es do not land next to each other is

$$\frac{7!}{2!} - 6! = 6!(7/2 - 1) = \frac{5}{2} \cdot 6! = 1800$$

Method 2: If we first remove the Es, there are 5! distinguishable arrangements of the remaining letters BYONC. Then, once we have have the arrangements of those 5 letters, we have $\binom{6}{2}$ placements for the Es where the do not land next to each other:

__B__Y__O__N__C__

Therefore there are

$$5!\binom{6}{2} = 1800$$

distinguishable arrangements where the Es are not next to each other.

(b) (Jay-Z is lending g's...) Jay-Z has \$10 which he is going to distribute amongst his 3 friends. How many ways can he distribute the money among them if each is to receive at least \$1? We assume here that he will distribute in dollar increments.

Solution: This is a direct application of finding *positive* integer solutions, where we are dividing up $n = 10$ indistinguishable objects into $k = 3$ bins. Therefore, Jay-Z can distribute the money in

$$\binom{10-1}{3-1} = \binom{9}{2}$$

ways.

10. An alien classroom has 29 students: 17 glargons and 12 bubuus. A committee of 2 students is to be selected.

(a) For $k = 0, 1$, or 2, what is the number of ways to choose this committee if k of the students are to be bubuus? Note: You can either write out for each case $k = 0$, $k = 1$, and $k = 2$ explicitly, or you can instead just write one general formula in terms of k.

Solution: For $k = 0, 1, 2$ there are $\binom{12}{k}$ ways to pick the k bubuus and $\binom{17}{2-k}$ ways to choose the glargons. Therefore, for each k we have

$$\binom{17}{2-k}\binom{12}{k}$$

ways to pick the committee of 2 with k bubuus.

(b) Suppose that each possible committee is equally likely. Let X be (the random variable giving) the number of bubuus selected for the committee. Find the probability mass function p_X of X.

Solution: Since each of the $\binom{29}{2}$ possible committees is equally likely, we have that for $k = 0, 1, 2$

$$\begin{aligned} p_X(k) = P(X = k) &= \frac{\text{\# of ways to pick committee of 2 with } k \text{ bubuus}}{\text{\# of ways to pick committee of 2}} \\ &= \frac{\binom{17}{2-k}\binom{12}{k}}{\binom{29}{2}} \end{aligned}$$

We could also represent this as a table

k	$p_X(k)$
0	$\binom{17}{2}/\binom{29}{2}$
1	$\binom{17}{1}\binom{12}{1}/\binom{29}{2}$
2	$\binom{12}{2}/\binom{29}{2}$

11. You are lost wandering through a desert with no water and you come upon the a lovely sorceress named Beyonce. Beyonce has 3 bottles in a box to her left: Bottle #1 containing 1 liter of water, bottle #2 containing 2 liters of water, and bottle #3 containing 3 liters of water. Beyonce will ask you a first question which you have a 50% chance of getting correct. If you get the answer wrong then she disappears along with all her water bottles and you get nothing. If you get the

answer correct, then you are given bottle #1 and she will ask you a second question which you have a 30% chance of getting correct. If you get the second question wrong then she disappears along with her remaining two bottles (you are left with only bottle #1 which you won in the first round). If you get the second question correct, then she gives you bottle #2 (so you now have bottles #1 and #2) and asks you a third question which you have a 10% chance of getting correct. If you get the third question wrong, then she disappears and takes the third bottle of water with her (you are left with bottles #1 and #2 which you won during the first two rounds). If you get the third question correct, then she gives you bottle #3 (so you now have bottles #1, #2, and #3) and she vanishes.

(a) Let X be (the random variable giving) the *liters of water* that you receive in your encounter with Beyonce. What is a good and reasonable state space S_X for the random variable X?

Solution: You have the possibility of getting the first question wrong, which will result in you receiving 0 liters of water. If you get the first question correct but miss the next you will have received 1 liter of water. If you get the first two questions correct but miss the third, then you will have 1 liter from the first bottle and 2 more from the second resulting in a total of 3 liters of water. Finally, if you get all questions correct, you will have 1 liter from the first bottle, 2 from the second, and 3 from the third, totaling 6 liters of water. So, a good and reasonable state space for X (which outputs the liters of water you receive) is

$$S_X = \{0, 1, 3, 6\}$$

(b) Let E be the event that you received bottles #1 and #2, but not bottle #3. Find $P(E)$.

Solution: Let R_i be the event that you received bottle i (for $i = 1, 2, 3$). Then what we want is

$$P(R_1 \cap R_2 \cap R_3^c).$$

We must use the multiplication rule here since R_1, R_2, and R_3 are certainly not independent. Hence

$$P(R_1 \cap R_2 \cap R_3^c) = \overbrace{P(R_1)}^{.5}\overbrace{P(R_2 \mid R_1)}^{.3}\overbrace{P(R_3^c \mid R_2 \cap R_1)}^{(1-.1)} = (.5)(.3)(.9) = .135$$

(c) Let k be the value in the state space S_X such that $\{X = k\}$ is the same event as E (where X and E are the same as the in the previous two parts). What is k?

Solution: We receive 3 liters of water if event E occurs. So $k = 3$ is the value we're looking for. That is $E = \{X = 3\}$.

12. Hogwarts School was invited to play at an international quidditch competition. At the school, Hogwarts has 12 Chasers, 8 Beaters, 4 Keepers, and 4 Seekers of which to choose 7 players (3 Chasers, 2 Beaters, 1 Keeper, and 1 Seeker) for the all-star team. Note that the team of 7 players will be picked in such a way that no chosen player changes position (e.g. a Chaser can only be a Chaser on the team). Assume that Lily is one of the four Seekers and her brother Albus is one of the twelve Chasers.

(a) How many different all-star teams can be formed where Lily is picked as Seeker?

Solution: We want to know the number of ways to select 3 Chasers from 12 possible Chasers, 2 Beaters from 8 possible Beaters, 1 Keeper from 4 possible Keepers, and pick Lily as Seeker. Note that there is only one way to choose Lily as Seeker.

$$\overbrace{\binom{12}{3}}^{Chasers} \cdot \overbrace{\binom{8}{2}}^{Beaters} \cdot \overbrace{\binom{4}{1}}^{Keeper} \cdot \overbrace{\binom{1}{1}}^{Lily}$$

(b) Suppose Lily is fighting with her brother Albus, and they refuse to play on the same team together. How many all-star teams can be formed such that Lily and Albus do not both end up being chosen? (Note that it is possible that Lily is not picked as the Seeker in this scenario).

Solution: Let E be the event that both Lily and Albus cannot be on the same team. First, let's find $\#E =$ the number of ways that both Lily and Albus cannot be on the same team. Since Lily and Albus cannot be on the same team, either Lily is on the team (in which case Albus is not), or Lily is not on the team. If Lily is on the team, then there are 11 remaining Chasers not including Albus from which I must choose 3. If Lily is not on the team, then there are 12 remaining Chasers from which I must choose 3. In either case, I must choose 1 Keeper from the 4 and 2 Beaters from the 8. Therefore, there are

$$\overbrace{\binom{11}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{1}{1}}^{\text{Lily on team}} + \overbrace{\binom{12}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{3}{1}}^{\text{Lily not on team}}.$$

Alternatively, we could find the same answer by doing something similar with Albus.

$$\overbrace{\binom{11}{2} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{3}{1}}^{\text{Albus on team}} + \overbrace{\binom{11}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}^{\text{Albus not on team}}.$$

A third way to find the number of ways is:

$$\overbrace{\binom{11}{2} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{3}{1}}^{\text{Albus on team, no Lily}} + \overbrace{\binom{11}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{1}{1}}^{\text{Lily on team, no Albus}} + \overbrace{\binom{11}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{3}{1}}^{\text{No Albus, no Lily on team}}.$$

A fourth way to calculate is:

$$\overbrace{\binom{12}{3} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}^{\text{All possible way to pick a team}} - \overbrace{\binom{11}{2} \cdot \binom{8}{2} \cdot \binom{4}{1} \cdot \binom{1}{1}}^{\text{Albus and Lily on same team}}.$$

13. Little Red is trying to get to her grandmother's house, but the Wolf is trying to stop her! To get to her grandmother's house, Little Red walks down a road until the road splits. The split forces her to either go left or right. If she goes to the left she must *independently* find either a cloak of invisibility or a sword (or both) to get past the Wolf and get to her grandmother's house. If she goes right she must find the Wolf-Off! spray to get past the Wolf and get to her

grandmother's house.

Assume that Little Red has a probability p_L of going left at the split in the road, and p_R of going right. Given that Little Red goes left at the split, she has a probability p_C to find the cloak of invisibility and probability p_S to find the sword. Given that she goes right at the split, then she has a probability of p_W of finding the Wolf-Off! spray.

Question: What is the probability that Little Red makes it to her grandmother's house?

Solution: Let G be the event that Little Red makes it to her grandmother's house. Let L be the event that she goes left at the fork, and R be the event that she goes right at the fork.

Suppose that Little Red goes left. Let C be the event that Little Red finds the invisibility cloak, let S be the event that she finds the sword. Then,

$$\begin{aligned}\overbrace{P(G|L)}^{\text{Red makes it going left}} &= \overbrace{P(C \cup S|L)}^{\text{finds the cloak or sword}} \\ &= P(C|L) + P(S|L) - P(C \cap S|L) \quad \text{by inclusion-exclusion} \\ &= P(C|L) + P(S|L) - P(C|L)P(S|L) \quad \text{by independence} \\ &= p_C + p_S - p_C p_S.\end{aligned}$$

That is to say, given that Little Red goes left, the probability she makes it to her grandmother's house is $p_C + p_S - p_C p_S$.

Otherwise, suppose she goes right. Let W be the event that she finds the Wolf-Off. Then,

$$\underbrace{P(G|R)}_{\text{makes it going right}} = \underbrace{P(W|R)}_{\text{finds the Wolf-Off}} = p_W.$$

Therefore,

$$P(G) = P(G|L)P(L) + P(G|R)P(R) = \boxed{(p_C + p_S - p_C p_S)p_L + p_W p_R.}$$

14. The following parts refer to the letters: LAMEFIREALARM. Recall that "word" means *distinguishable* letter arrangements.

    (a) How many words can be made with the above letters such that the M's are *not* next to each other?

    Solution: There are 13 total letters: 2 L's, 3 A's, 2 M's, 2 E's, 1 F, 1 I, and 2 R's. Using the word counting principle, this means there are

    $$\binom{13}{2,3,2,2,1,1,2} = \binom{13}{3,2,2,2,2} = \frac{13!}{3!2!2!2!2!} = 64\,864\,800$$

    total possible words. Of these, by grouping together the 2 M's, we see there are

    $$\binom{12}{2,3,1,2,1,1,2} = \binom{12}{3,2,2,2} = \frac{12!}{3!2!2!2!} = 9\,979\,200$$

    words in which both M's *are* next to each other. Therefore there are

    $$\boxed{\binom{13}{3,2,2,2,2} - \binom{12}{3,2,2,2} = 54\,885\,600}$$

words in which the M's are not next to each other.

(b) What is the probability that, given a random word made from the letters above, the M's will be separated by at least two letters?

Solution: We will start similar to the last part, where we will find the total number of words where the M's are separated by at least two letters by taking the total number of words and subtracting off the number of words where the M's are together and also subtracting off the number of words where the M's are separated by 1 letter. To find the number of words where the M's are separated by 1 letter, notice that if we fix the position of the M's, there are $\binom{11}{3,2,2,2}$ distinguishable arrangements of the other 11 letters. There are 11 different possible positions for the M's such that they are separated by exactly one letter, so there are a total of $11 \times \binom{11}{3,2,2,2}$ words where the M's are exactly 1 letter apart. Hence the number of words where the M's are at least two letters apart is:

$$\underbrace{\overbrace{\binom{13}{3,2,2,2,2}}^{\text{all words}} - \overbrace{\binom{12}{3,2,2,2}}^{\text{M's together}}}_{\text{from part (a)}} - \overbrace{11 \times \binom{11}{3,2,2,2}}^{\text{M's separated by 1 letter}} = 45\,738\,000$$

We now only need to divide this by the total number of words to get the probability:

$$\boxed{\frac{\binom{13}{3,2,2,2,2} - \binom{12}{3,2,2,2} - 11 \times \binom{11}{3,2,2,2}}{\binom{13}{3,2,2,2,2}} = \frac{55}{78} \approx 70.5\%}$$

Note: Let's make sure I'm clear about how the counting went in this part. To find the number of words where the M's were separated by exactly one letter, here is the idea: Let $*$ represent the non-M letters. Then the words must look like:

$$11 \text{ positions for the 2 M's} = \begin{cases} \underline{M}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*} \\ \underline{*}\,\underline{M}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*} \\ \vdots \\ \underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{*}\,\underline{M}\,\underline{*}\,\underline{M} \end{cases}$$

However, once we fix where the two M's are, the other 11 letters (represented above by $*$) can be rearranged in $\binom{11}{3,2,2,2}$ distinguishable ways by the word counting principle. The principle of counting then says there are $11 \times \binom{11}{3,2,2,2}$ total such words.

15. Suppose you shuffle together two standard 52 card decks (effectively, you double the number of each card). You are randomly dealt a 5 card poker hand. Let X be the random variable counting the number of 2's in your hand.

(a) What is the state space S_X of X?

Solution: Since there are eight 2's now, we can have anywhere from 0 to 5 in our hand. So,

$$\boxed{S_X = \{0, 1, 2, 3, 4, 5\}}$$

(b) What is the probability mass function p_X of X? That is, find $p_X(k) = P(X = k)$ for k in S_X.

Solution: For each k in S_X, we have $\binom{8}{k}$ ways to choose the 2's that go in our hand, $\binom{96}{5-k}$ choices for the other non-2's that go in our hand. There are a total of $\binom{104}{5}$ poker hands from the doubled deck so,

$$\boxed{p_X(k) = P(X = k) = \frac{\binom{8}{k}\binom{96}{5-k}}{\binom{104}{5}}}$$

(c) What is the value of the distribution F_X of X at 5.1? That is, find $F_X(5.1) = P(X \leq 5.1)$.

Solution: Since the maximum value of X is 5, the probability that $X \leq 5.1$ (or any number ≥ 5 in fact!) is 1. In other words,

$$\boxed{F_X(5.1) = P(X \leq 5.1) = 1}$$

16. Marble 1, marble 2, and marble 3 are in an urn. You select one of the three marbles out of the urn in such a way that you have a 40% chance of selecting marble 1, a 50% chance of selecting marble 2, and a measly 10% chance of selecting marble 3. After you've selected the marble you then roll fair dice in the following way: if you selected marble 1, you roll one fair die; if you selected marble 2, you roll two fair dice; if you selected marble 3, you roll three fair dice. Given that the sum of the outcomes of the dice you roll is 3, what is the probability that you selected marble 2? (To make sure we're clear: If you roll three dice and the outcomes are i, j, and k, then the sum of the outcomes is $i + j + k$. If you roll two dice and the outcomes are i and j, then the sum of the outcomes is $i + j$. If you roll one die and the outcome is i, then the sum of the outcomes is i.)

Solution: Let M_i be the event that you selected the i^{th} marble ($i = 1, 2, 3$) and let T be the event that the sum of the outcomes you rolled is 3. We want to find $P(M_2|T)$; we will use Bayes' theorem to do so. We can easily find $P(T|M_1) = 1/6$ since we have a 1/6 probability of rolling 3 with one die. Similarly $P(T|M_2) = 2/6^2 = 2/36$ since there are 2 out of 36 possible outcomes of rolling 2 dice where the sum is 3 (the outcome (1,2) or (2,1)). Also $P(T|M_3) = 1/6^3 = 1/216$ since there is only 1 out of the 216 possible outcomes of rolling 3 dice where the sum is 3 (the outcome (1,1,1)). Hence,

$$\begin{aligned} P(M_2|T) &= \frac{P(T|M_2)P(M_2)}{P(T|M_1)P(M_1) + P(T|M_2)P(M_2) + P(T|M_3)P(M_3)} \quad \text{by Bayes'} \\ &= \boxed{\frac{(2/36)(.5)}{(1/6)(.4) + (2/36)(.5) + (1/216)(.1)} = \frac{12}{41} \approx 29.3\%} \end{aligned}$$

17. You play a game where during each round you win either \$1 with probability 2/3 or you win \$3 with probability 1/3. Assume that the outcome of each round is independent of the outcomes

of any other rounds. Your strategy is that you will play until the first time your winnings total at least \$4 and then you will quit. For example, if on the first two rounds you won \$1 and the third round you won \$3, then you will quit since your total winnings are \$5. Let X be the random variable representing your total winnings when you quit playing.

(a) What is the state space of X?

Solution: We have several ways to exit the game, and we'd like to know all the possible amounts of money we could win when we quit. We have the following scenarios:

$$\Omega = \{(1,1,1,1),(1,3),(3,1),(1,1,3),(1,1,1,3),(3,3)\}$$

where $(1,1,1,1)$ represents getting \$1 on round 1, \$1 on round 2, \$1 on round 3, and \$1 on round 4. Then, X can be \$ 4 (three ways to do this), \$ 5 (one way to do this), or \$6 (2 ways to do this). Thus,

$$S_X = \{4,5,6\}$$

(b) Find $P(X = 4)$.

Solution: From above, we know that there are 3 ways to get \$4. Thus,

$$P(X=4) = \overbrace{\left(\frac{2}{3}\right)^4}^{(1,1,1,1)} + \overbrace{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}^{(1,3)} + \overbrace{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}^{(3,1)} = \frac{52}{81}$$

(c) If F_X is the cumulative distribution function of X, what is $F_X(4.5)$?

Solution: Recall that $F_X(4.5) = P(X \leq 4.5)$. In our case, $P(X \leq 4.5) = P(X = 4)$. Thus, from Part (b)

$$F_X(4.5) = P(X \leq 4.5) = P(X = 4) = \frac{52}{81}.$$

(d) If N is the random variable representing the number of rounds you played before quitting, briefly explain why $P(X = 5 \mid N = 2) = 0$.

Solution: If we played two rounds ($N = 2$) and quit, we know from our work in Part (a) that either $(1,3)$, $(3,1)$ or $(3,3)$ occurred. That is, the only possible outcomes of X are $X = 4$ or $X = 6$. Thus, $P(X = 5 \mid N = 2) = 0$.

18. Your jar of Bertie Bott's Every Flavour Beans contains only 2 flavors: 8 flavored like Strawberry and Peanut-Butter Ice Cream (SPB) and 4 flavored like Lobster (L). You play the following game with your friend James (who loves SPB): If James draw a SPB, he eats it and you add 3 L to the jar. If James draws an L, you eat it and you add 1 SPB to the jar.

(a) What is the probability that James draws 3 SPB in a row?

Solution: Let SPB_i be the event that SPB was picked on the ith draw and let L_i be the event that L was picked on the ith draw. We want to find $P(SPB_1 \cap SPB_2 \cap SPB_3)$. By the Multiplication Rule,

$$P(SPB_1 \cap SPB_2 \cap SPB_3) = P(SPB_1)P(SPB_2 \mid SPB_1)P(SPB_3 \mid SPB_1 \cap SPB_2)$$

In our jar of beans, we have a total of 12 beans to start: 8 SPB and 4 L. Thus, on our first draw

$$P(SPB_1) = \frac{8}{12}.$$

After the first draw, our bean count changes. We now have $8 - 1 = 7$ SPB, $4 + 3 = 7$ L, and a total number of 14 beans. Then,

$$P(SPB_2 \mid SPB_1) = \frac{7}{14}.$$

After the second draw, our bean count changes. We now have $7 - 1 = 6$ SPB, $7 + 3 = 10$ L, and a total number of 16 beans. Then,

$$P(SPB_3 \mid SPB_1 \cap SPB_2) = \frac{6}{16}.$$

So,

$$P(SPB_1 \cap SPB_2 \cap SPB_3) = \left(\frac{8}{12}\right)\left(\frac{7}{14}\right)\left(\frac{6}{16}\right) = \frac{1}{8}.$$

(b) What is the probability that on James' second draw, he draws L?

Solution: We are drawing two beans, and on the first draw we could either draw a SPB or a L. We want the probability that James draws L on the second draw. So,

$$\begin{aligned} P(L_2) &= P((SPB_1 \cap L_2) \cup (L_1 \cap L_2)) \\ &= P(SPB_1 \cap L_2) + P(L_1 \cap L_2) \\ &= P(SPB_1)P(L_2 \mid SPB_1) + P(L_1)P(L_2 \mid L_1) \end{aligned}$$

From Part (a), we know that $P(SPB_1) = \frac{8}{12}$, and we know that after that draw we have 7 SPB and 7 L. Then,

$$P(L_2 \mid SPB_1) = \frac{7}{14}.$$

From Part (a), we also know that $P(L_1) = \frac{4}{12}$. After that first draw, we would have $4 - 1 = 3$ L and $8 + 1 = 9$ SPB with a total of 12 beans. Then,

$$P(L_2 \mid L_1) = \frac{3}{12}.$$

Therefore,

$$\begin{aligned} P(L_2) &= P(SPB_1)P(L_2 \mid SPB_1) + P(L_1)P(L_2 \mid L_1) \\ &= \left(\frac{8}{12}\right)\left(\frac{7}{14}\right) + \left(\frac{4}{12}\right)\left(\frac{3}{12}\right) \\ &= \frac{5}{12} \end{aligned}$$