PRACTICE PROBLEMS FOR THE FINAL

Math 3160Q – Fall 2015
Professor Hohn

Below is a list of practice questions for the Final Exam. I would suggest also going over the practice problems and exams for Exam 1 and Exam 2 to help you prepare. Any quiz, homework, worksheet, or example problem has a chance of being on the exam. For more practice, I suggest you work through the review questions at the end of each chapter as well.

1. The amount of time a customer spends at a certain store is modeled by an exponential random variable with mean 10 minutes. If each customer's time at the store is independent, use the Central Limit Theorem to approximate the probability that 100 randomly selected customers spend between 950 and 1050 minutes at the store. Leave your answer in terms of the standard normal distribution $\Phi(x)$.

2. Let X and Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} a\,(x+y)\,e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the constant a. (Yes, you will need integration by parts.)

   (b) Set up the integral you would use to calculate $P(2X < Y)$. Make sure to clearly indicate the bounds on the integral. You do not need to evaluate the integral.

   (c) Are X and Y independent? Make sure you justify your answer.

3. This problem shows that not all functions can be moment generating functions, even if they're very differentiable near zero! Let $f(t) = e^t + t$.

   (a) Pretend that $f(t)$ is the moment generating function for some random variable X and use it to find $\mathrm{Var}(X)$.

   (b) Using the value of $\mathrm{Var}(X)$ from (a), how do you know that no such X can exist?

4. Suppose the number of meteors hitting earth is modeled by a Poisson process $(N_t)_{t\geq 0}$ with parameter $\lambda = 5$ (meteors / hour) where t is in hours.

   (a) What is the probability that during the 1 hour you are in this class, 2 or more meteors hit the earth? I don't want you to leave me with an infinite sum, instead it should be something that you could calculate easily *if* you were allowed a calculator.

   (b) What is the conditional probability that during the 1 hour you are in this class, 2 or more meteors hit the earth given that 1 or more meteors hit the earth?

5. Suppose X is a continuous random variable with distribution function given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

(a) Find the density $f_X(x)$ of X.

(b) Find $\mathrm{Var}(X)$.

6. The following parts (a) and (b) refer to the letters MMMMPPIII.

   (a) How many words can be created such that both P's occur to the left of the first I?

   (b) Suppose each word is equally likely to occur. What is the probability that you randomly select a word where both P's occur to the left of the first I?

7. Suppose that there is a disease within a population. Each individual within the population independently and randomly has the disease with probability p. A certain test is created to screen individuals for the disease. If an individual has the disease, there is a probability q that the test correctly reports positive. If an individual does not have the disease, there is a probability r that the test falsely reports positive for the disease. If a randomly selected individual tested positive for the disease, what is the probability that they have the disease? You should assume that the test only reports a positive or negative result, and your answer should be in terms of p, q, and r.

8. Your "friend" wants to play a gambling game with you. The game goes as follows: You pay your friend \$$C$ to play. You then draw twice from an urn initially containing 4 red marbles and 5 black marbles. On the first draw you take a single marble and return it back into the urn with another marble of the same color. On the second draw you pull out a single marble: If it is a red marble you win \$5, whereas if it is a black marble you win nothing.

   Question: What is the maximum amount of money \$$C$ you should pay your friend to play so that on average you don't lose money?

9. You must select exactly one of two challenges: A or B. If you select challenge A you are forced to answer a question which you have a $1/2$ probability of getting correct. If you answer correctly you will be rewarded silver. If you answer incorrectly, you will be forced to eat spoiled cheese. If you select challenge B, you will be forced to complete a physical task which you have $1/3$ chance of completing. If you succeed you will rewarded gold. If you do not succeed you will be forced to eat spoiled cheese. You decide to flip a biased coin with probability $2/3$ of getting heads to decide which challenge to take: heads means you take A, tails means you take B. Given that you are forced to eat spoiled cheese, what is the probability that you selected challenge A?

10. Each year, the cryptids known only as "tubes" independently kill Americans with an average kill rate of 3 Americans per year.

    (a) Using a Poisson distribution, approximate the probability that 2 or fewer Americans die next year from "tubes."

    (b) Briefly explain why a Poisson approximation here is a reasonable choice and why one might use it over the Central Limit Theorem (the normal approximation)?

11. Let X and Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} c\,(1-y) & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

    (a) Find c.

(b) Given that $Y > 2/3$, what is the probability that $X < 1/2$?

12. Let X be a random variable with distribution

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2}x & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

(a) Is X continuous, discrete, or neither? Remember to give me a brief justification.

(b) Find $\mathbb{E}[X]$.

13. You roll two fair six-sided dice. Let X be the minimum value of the two rolls showing. For example, if you roll a 3 and a 4, then X is 3, since $\min(3,4) = 3$. Similarly, if you roll a 5 and a 5, then X is 5, since $\min(5,5) = 5$.

(a) What is a reasonable sample space Ω for this experiment?

(b) What is the state space S_X of X?

(c) Find the probability mass function p_X of X.

14. Carbon-14 has a half-life of approximately 5730 years. Organic life continually replenishes its stock of carbon-14 until death at which point the carbon-14 decays without replenishing. An organism with about 6000 carbon-14 particles just died. Let N_t count the number of decayed carbon-14 particles after a time t of the organism's death (where t is measured in years). Based on the information just described, it is reasonable to model $(N_t)_{t \geq 0}$ as a Poisson process with parameter $\lambda = .725$.

(a) Assuming that $(N_t)_{t \geq 0}$ is a Poisson process with parameter $\lambda = .725$, what is the probability that two or more carbon-14 particles decay during one year following the death of the organism? Please don't leave your answer as an infinite sum!

(b) Assuming that $(N_t)_{t \geq 0}$ is a Poisson process with parameter $\lambda = .725$, what is the probability that no carbon-14 particles decay during the first year following the organism's death, *and* two carbon-14 particles decay during the following two years?

15. Suppose that the moment generating function for a random variable X is given by

$$M_X(t) = \frac{1}{(1-t)^2}$$

(a) Find a formula for the general n^{th} moment of X. That is, in terms of n, find $\mathbb{E}[X^n]$.

(b) Find $\mathrm{Var}(X)$.

16. Suppose that $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Fix some $\varepsilon > 0$. Use the Central Limit Theorem to approximate

$$P\Big(-\varepsilon < \frac{1}{n}\sum_{i=1}^{n} X_i - \mu < \varepsilon\Big).$$

Leave your answer in terms of the standard normal distribution $\Phi(a) = P(N(0,1) \leq a)$. Your answer should depend on ε, n, and σ^2.

Hint: $\displaystyle \frac{1}{n}\sum_{i=1}^{n} X_i - \mu = \frac{1}{n}\left(\sum_{i=1}^{n} X_i - n\mu\right).$

17. Let X be a random variable with cumulative distribution function

$$F_X(s) = \begin{cases} 0 & s < 1 \\ \frac{1}{2}\, s & 1 \leq s < 2 \\ 1 & 2 \leq s \end{cases}$$

(a) Let Y be the random variable defined by $Y = \ln(X)$ (here, ln is the natural log). Find the cumulative distribution function F_Y of Y.

(b) Find $\mathbb{E}[\ln(X)]$.

18. Let X and Y be jointly discrete random variable with probability mass function $p_{X,Y}(s,t)$ described by the following table

X \ Y	0	1	2
1	$\frac{1}{2}$	0	$\frac{1}{4}$
4	0	$\frac{1}{8}$	0
9	$\frac{1}{16}$	0	$\frac{1}{16}$

(a) What is the state space S_X of X and the state space S_Y of Y?

(b) Calculate $\mathbb{E}[\sqrt{X}\,Y]$.

(c) Are X and Y independent? Make sure to justify your answer.

19. The microorganism E.coli living in ***your*** body has the average lifespan of 10 hours. Right at this moment you, the E.coli whisperer, befriend one of these living E.coli.

(a) If the lifespan of the E.coli is given by an exponential random variable, what is the probability that your E.coli friend is still alive in five hours from now? If you are unable to answer this question, make sure to give a good and brief explanation why you can't answer it and what more information you would need to be be able to answer.

(b) If the lifespan of the E.coli is uniformly distributed between 0 and 20 hours (i.e., has a $\mathrm{Unif}(0, 20)$ distribution), what is the probability that your E.coli friend is still alive in five hours from now? If you are unable to answer this question, make sure to give a good and brief explanation why you can't answer it and what more information you would need to be be able to answer.

20. A real number is chosen uniformly from $[0, 3]$ and then this number is squared. Let X represent the result. (That is, if U is the number chosen from $[0, 3]$, $X = U^2$.)

(a) What is the cumulative distribution function of X?

(b) What is the density of X?

21. Tired of Candy Crush, you decide to play a new game. The game goes as follows: Initially there are two marbles in an urn; one marble is red, the other is green. At each round of the game, you will randomly select a marble, then put the marble back into the urn along with another marble of the opposite color (e.g., if you draw a red marble, you will put the red marble back into the urn along with a green marble). The game is over when either you draw a

green marble, or you have played for three rounds without drawing a green marble (whichever happens first). Let X be the random variable representing the number of rounds you play until the game ends.

(a) What is the state space of X?

(b) What is the probability mass function of X?

(c) What is the expected value of X?

22. Following her true dream, your professor opens a surf shop (and school) in Puerto Rico. Suppose that for each i, we let X_i be the number of customers that make a purchase at your professor's shop on day i. Assume that the collection X_1, X_2, $X_3, \ldots$ are independent and identically distributed such that $X_i \overset{d}{=} \text{Pois}(1)$. Use the Central Limit Theorem (i.e., a normal approximation) to approximate the probability that at most 7 customers made a purchase within the first 9 days. That is, approximate $P\big(\sum_{i=1}^{9} X_i \leq 7\big)$ using the Central Limit Theorem. You can leave your answer in terms of Φ, the CDF of a standard normal random variable.